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On classifying monotone complete algebras of operators

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Abstract We give a classification of "small" monotone complete C^*-algebras by order properties. We construct a corresponding semigroup. This classification filters out von Neumann algebras; they are mapped to the zero of the classifying semigroup. We show that there are 2^c distinct equivalence classes (where c is the cardinality of the continuum). This remains true when the classification is restricted to special classes of monotone complete C^*-algebras e.g. factors, injective factors, injective operator systems and commutative algebras which are subalgebras of ℓ^∞. Some examples and applications are given.

Keywords Monotone complete C^*-algebras · Operator algebras · Semilattices

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1 Introduction

Let A be a C^*-algebra then its self-adjoint part A_{sa} has a natural partial ordering. If each norm bounded, upward directed subset of A_{sa} has a least upper bound then A is said to be *monotone complete*; when A is monotone complete it possesses a unit. Every von Neumann algebra is monotone complete but the converse is false [5].

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Monotone complete C^*-algebras arise in a number of different areas. For example, injective operator systems can always be given the structure of a monotone complete C^*-algebra and, in the theory of operator spaces, the injective operator spaces can be realised as "corners" of monotone complete C^*-algebras. [7, Theorem 6.1.3 and Theorem 6.1.6] (see also [11]) When A is a commutative monotone complete C^*-algebra then its lattice of projections is a complete Boolean algebra. Conversely, every complete Boolean algebra has a unique (up to isomorphism) representation of this form. So the study of commutative monotone complete C^*-algebras is equivalent to the study of complete Boolean algebras. ([10] and [5]. See also [3]).

Let A be a monotone complete C^*-algebra. It is said to be a *factor* if its centre is one dimensional. Intuitively speaking, a factor is as far removed from being commutative as possible. Just as for von Neumann algebras, monotone complete factors can be divided into Type I, Type II_1, Type II_∞ and Type III. (See for example [3]) An old result of Kaplansky [18] implies that every Type I factor is a von Neumann algebra. Kaplansky's result made it natural for him to ask if all factors were von Neumann algebras. The answer is "no" but to make this intelligible the following definitions are helpful.

First let us recall that a C^*-algebra A is said to be *separably representable* if there exists an (isometric) $*$-isomorphism π from A into $L(H)$, the algebra of all bounded operators on a separable Hilbert space H.

A (unital) C^*-algebra A is said to be *small* if there is a unital, complete isometry from A into $L(H)$, where H is separable. An excellent account of operator spaces, operator systems, complete isometries, completely positive maps and their properties is given by Effros and Ruan [7]. When ϕ is a unital complete isometry of B into $L(H)$ then ϕ is a completely positive isometry onto an operator system, and its inverse is completely positive. [7, Corollary 5.1.2]

Some thirty years ago Wright [40] showed that if a monotone complete factor A is separably representable then it is a von Neumann algebra. (See also [28]). Further results of Wright [37] also imply that if a Type II factor is small then it is also a von Neumann algebra. (See also [4], [8] and [25]). So if a monotone factor is small and is not a von Neumann algebra then it must be of Type III. (A factor which is not a von Neumann algebra is called *wild.*) Finding examples of these wild Type III factors was elusive. The first examples were due, independently, to Dyer [6] and Takenouchi [32] and in [25], Saitô showed that both factors are of Type III; later, as a consequence of a more general result [31], their factors were shown to be isomorphic. Wright also gave a general construction and showed that the regular σ-completion of any unital, simple separably representable infinite dimensional C^*-algebra was a Type III factor which is never a von Neumann algebra ([38] and [39], see also [22]). For many years only a few examples were known to be non-isomorphic (see [14], [26] and for large algebras, see [27]), until a big breakthrough by Hamana [15]. He used delicate cardinality arguments to show that there are a huge number of mutually non-isomorphic Type III factors which are not von Neumann algebras but which are small C^*-algebras. (In this context "huge" means 2^c where c is the cardinality of the real numbers). This paper by Hamana is not yet as widely known as it deserves to be. It is fundamental for our work here. In turn, his work (like ours) makes key use of ideas of Monk and Solovay on complete Boolean algebras [21].

In the first section of this paper we define a quasi ordering between monotone complete C^*-algebras and use this to obtain an equivalence relation. Roughly speaking, ignoring some set theoretic technicalities for the moment, the equivalence classes of small monotone complete algebras can be organised into a partially ordered, abelian semigroup, W; where taking direct sums of algebras corresponds to the semigroup addition. Furthermore, we prove that the semigroup has the Riesz decomposition property. Its zero corresponds to the von Neumann algebras. Influenced by K-theory, a natural response to a non-cancellative, abelian semigroup is to form its Grothendieck group; this is useless for W since its Grothendieck group is trivial. This is because every element of the semigroup W is idempotent. By a known general theory [9] this idempotent property implies that the semigroup can be identified with a join semilattice; then the Riesz decomposition property is equivalent to the semilattice being distributive. This means that the well established theory of distributive, join semilattices can be applied to W.

The following notation is convenient. For each monotone complete C^*- algebra A there corresponds an element, $w(A)$, in W such that $w(A) = w(B)$ precisely when A and B are equivalent. We call $w(A)$ the *normality weight* of A and W the *weight* semigroup.

One feature of this classification theory is that some problems involving factors can be replaced by problems involving commutative algebras. For example, let A_j $(j = 1,2)$ be commutative monotone complete C^*- algebras; let G_j $(j = 1,2)$ be countable discrete groups with free, ergodic actions on, respectively, A_j $(j = 1,2)$. Then, by a cross product construction using these group actions, we can construct monotone complete C^*-factors B_j $(j = 1,2)$. It turns out to be easy to show that $w(B_j) = w(A_j)$ for $j = 1,2$. Now suppose we know that $w(A_1) = w(A_2)$. Then the factors B_1 and B_2 must have the same normality weight. In particular, when we apply the cross product construction to inequivalent commutative algebras then the factors constructed cannot be isomorphic.

For each monotone complete C^*-algebra, A, we define a spectroid, ∂A. It turns out that if A and B have the same normality weight then $\partial A = \partial B$. So the spectroid is an invariant for the elements of the weight semigroup. (To be more precise there is a family of spectroids. On fixing a parameter set T and an injective map $\mathbf{N}$ from T into the collection of infinite subsets of $\mathbb{N}$, we can obtain a corresponding spectroid for each A).

We know that the classification classes can contain very diverse algebras. For example, all small von Neumann algebras are equivalent to $\mathbb{C}$. On the other hand, the classification is sufficiently refined to distinguish between a huge number of small monotone complete algebras; the cardinality of the semigroup W is 2^c. (We make use of spectroids to show this). When restricted to subclasses (e.g. small factors, small injective factors, injective operator systems, commutative monotone complete algebras which are subalgebras of ℓ^∞) the normality weight classification still distinguishes between 2^c objects. Many possible generalisations and modifications of these constructs are possible; some of these will be indicated below. They will be presented in later work. In this paper we strive to maximise clarity rather than generality.

We modify the approach of Monk-Solovay [21] and of Hamana [15], to construct examples of small commutative monotone complete C^*-algebras. We show that these algebras take 2^c-distinct normality weights. We show that each of our examples is of the form $B^\infty(K)/J$ where K is the Cantor space, $\{0,1\}^{\mathbb{N}}$, and $B^\infty(K)$ is the algebra of bounded Borel functions on K and J is a σ-ideal of $B^\infty(K)$. Moreover each of these $B^\infty(K)/J$ can be embedded as a closed $*$-subalgebra of ℓ^∞. We can construct 2^c ideals such that, for every countably infinite discrete group G, G has a free, ergodic action on $B^\infty(K)/J$ and the weights of these algebras take 2^c distinct values. Such actions lead, by using cross products, to associated small wild factors.

We begin this construction by taking the product space $\{0,1\}^{\mathbb{R}}$. This is a compact Hausdorff space which, somewhat surprisingly, has a countable dense subset [16]. Let D_0 be any countable, infinite subset of $\{0,1\}^{\mathbb{R}}$. Then let D be the closure of D_0. Clearly D is a compact Hausdorff space with a countable dense subset, D_0. So there is a natural isometric $*$-isomorphism of $C(D)$ into ℓ^∞. By a general theory, the regular (σ-)completion of $C(D)$ can be identified with a commutative monotone complete algebra which can be embedded as a subalgebra of ℓ^∞. The structure space of each such commutative monotone complete algebra can be identified with a closed subspace of $\{0,1\}^{\mathbb{R}}$. By appropriate choices of the countable set D_0 we find the algebras we need.

Monotone complete C^*-algebras are a generalisation of von Neumann algebras. The theory of the latter is now very well developed. Major breakthroughs by McDuff [20] and Powers [24] constructed the first infinite collections of factors of, respectively, Type II_1 and Type III (see also Sakai [30], Araki-Woods [2], Krieger [19]). Then the work of Takesaki, Connes and the other giants of the subject transformed the theory and understanding of von Neumann algebras (see [34]). This theory was so powerful and so dominant that, for many years, people strived to imitate it for general monotone complete C^*-algebras. This had only mixed success. With the major advance of Hamana [15] as our starting point we give a classification which is totally different from the methods used in von Neumann algebra theory.

The following analogy may be helpful. Consider a vast city where each building contains a small monotone complete C^*-algebra and every algebra which is isomorphic to it. By Hamana's pioneering work [15], there are 2^c-buildings. Our classification splits the whole city into parallel avenues, running west to east. At the centre is the 0^{th} avenue, housing all the small von Neumann algebras. There are 2^cavenues. Intersecting the avenues are streets running north to south. One of these streets is that where all the small commutative algebras are to be found. Many other streets remain to be explored before a complete map of the city can be made. Nevertheless the classification given here helps to bring some order out of chaos.

2 Preliminaries

Let us recall that a C^*-algebra is monotone σ-complete if each upper bounded, monotone increasing sequence of self-adjoint elements has a supremum. Clearly all monotone complete C^*-algebras are monotone σ-complete but the converse is false. (For example the algebra of all bounded Borel functions on the unit interval is monotone σ-complete but not monotone complete).

Lemma 1 *[29] Let A be a monotone σ-complete C^*-algebra. Let there exist a positive linear functional $\mu : A \to \mathbb{C}$ which is faithful. Then A is monotone complete. Whenever Λ is a downward directed subset of A_{sa} which is bounded below, there exists a monotone decreasing sequence (x_n), with each $x_n \in \Lambda$, such that $\bigwedge_{n=1}^{\infty} x_n$ is the greatest lower bound of Λ.*

Corollary 1 *When a small C^*-algebra is monotone σ-complete then it is monotone complete.*

Proof Each small C^*-algebra, A, has a faithful state; because A has a positive linear injection into $L(H)$, where H is separable. □

Proposition 1 *Whenever A is a small C^*-algebra then its cardinality is $c = 2^{\aleph_0}$.*

Proof Because H has a countable orthonormal basis, each operator on H can be represented by an infinite $\mathbb{N} \times \mathbb{N}$ matrix over the complex numbers. So $L(H)$ injects into $\mathbb{C}^{\mathbb{N}\times\mathbb{N}}$ which has cardinality c. Since A can be injected into $L(H)$, it too has cardinality c. □

The following proposition is not essential for the work that follows so we omit the proof. For a definition of the Pedersen Borel envelope, see [23].

Proposition 2 *Let A be any C^*-algebra of cardinality c. Then $B^{\infty}(A)$, the Pedersen Borel Envelope of A, has cardinality c.*

Corollary 2 *Whenever A is a small C^*-algebra then its Pedersen Borel Envelope has cardinality c.*

We use $\#S$ to denote the cardinality of a set S.

Proposition 3 *Let A be any C^*-algebra of cardinality c. Then it has a faithful representation on a Hilbert space of cardinality c.*

Proof We may assume that the algebra is unital, if not we can adjoin a unit without increasing the cardinality.

For each $a \in A \backslash \{0\}$, there is a pure state ϕ_a such that $\phi_a(aa^*) \neq 0$. By the GNS process and the fact that the state is pure, there is a surjection from A onto the corresponding GNS Hilbert space $H(\phi_a)$. So $\#H(\phi_a) \leq \#A = c$. Let H be the Hilbert space direct sum of $\{H(\phi_a) : a \in A\backslash\{0\}\}$. So H has an orthonormal basis of cardinality not exceeding $c \times c = c$. Since each element of the Hilbert space is orthogonal to all but countably many basis elements, $\#H \leq c \times c^{\aleph_0} = c$. The natural representation of A on H is faithful. □

Remark 1 When A is a monotone σ-complete C^*-algebra and D is a downward directed set in A^+ which is countable and bounded below then there exists a monotone decreasing sequence (c_n) $(n = 1, 2, \dots)$ in D such that $\bigwedge_{n=1}^{\infty} c_n$ is a lower bound for D and hence the infimum of D. This is easily proved by an inductive construction of the sequence.

Remark 2 Whenever A is a monotone σ-complete C^*-algebra then there is a σ-homomorphism from its Pedersen Borel envelope, $B^{\infty}(A)$, onto A [36].

Remark 3 Let H be a separable infinite dimensional Hilbert space and let $\mathscr{V}$ be the collection of all von Neumann subalgebras of $L(H)$. Then $\#\mathscr{V} = c$.

Proposition 4 *Each small von Neumann algebra is separably representable as a von Neumann algebra subalgebra of $L(H)$ where H is a separable Hilbert space.*

Proof Let A be a small unital C^*-algebra. Then its state space can be identified with the state space of an operator system in $L(H)$, where H is a separable Hilbert space. By the Hahn-Banach Theorem this state space is the continuous image of the state space of $L(H)$ and hence is separable. Let A be a von Neumann algebra. Then, by a theorem of Akemann [1], A has a norm separable predual and hence A has a faithful normal separable representation. In other words, A can be identified with a von Neumann subalgebra of $L(H)$, where H is a separable Hilbert space. □

3 Classification semigroups

Our focus is on small monotone complete C^*-algebras but much of the work here can be done in much greater generality. In one direction, weight semigroups can be defined, with no extra difficulty, for monotone complete C^*-algebras of arbitrary size. To avoid some set theoretic difficulties we fix a large Hilbert space $H^\#$ and, for the rest of this section, only consider algebras which are isomorphic to subalgebras of $L(H^\#)$. For (unital) small C^*-algebras, their Pedersen Borel envelopes, or more generally any (unital) C^*-algebra of cardinality $c = 2^{\aleph_0}$, it suffices if $H^\#$ has an orthonormal basis of cardinality $c = 2^{\aleph_0}$. (See Section 2) We call the corresponding classification semigroup, W, the normality weight semigroup.

Let A and B be monotone complete C^*-algebras and let $\phi : A \to B$ be a positive linear map. We recall that ϕ is *faithful* if $x \geq 0$ and $\phi(x) = 0$ implies $x = 0$.

Definition 1 Let A and B be monotone complete C^*-algebras and let $\phi : A \to B$ be a positive linear map. Then ϕ is said to be normal if, whenever D is a downward directed set of positive elements of A, ϕ maps the infimum of D to the infimum of $\{\phi(d) : d \in D\}$.

When defining the classification semigroup we shall use positive linear maps which are faithful and normal. By varying the conditions on ϕ we get a slightly different theory. For example, we could strengthen the conditions on ϕ by also requiring it to be completely positive. Essentially all the construction in this section can also be carried out with completely positive maps. We then get a semigroup which has a natural quotient onto the semigroup W. On the other hand, we could weaken the conditions by (i) only requiring ϕ to be σ-normal and (ii) only requiring the algebras to be monotone σ-complete; we still obtain a classification semigroup but can no longer show that it has the Riesz decomposition property. When we restrict our attention to small C^*-algebras, we saw in Section 2 that monotone σ-complete implies monotone complete. So in this setting, normal and σ-normal maps coincide.

Let Ω be the class of all monotone complete C^*-algebras which are isomorphic to norm closed $*$-subalgebras of $L(H^\#)$; and let $\Omega^\#$ be the set of all C^*-subalgebras of $L(H^\#)$ which are monotone complete (in themselves, they cannot be monotone closed subalgebras of $L(H^\#)$ unless they are von Neumann algebras). So every $A \in \Omega$ is isomorphic to an algebra in $\Omega^\#$.

We define a relation on Ω by $A \precsim B$ if there exists a positive linear map $\phi : A \to B$ which is faithful and normal.

Let π be an isomorphism of A onto B. Then π and π^{-1} are both normal so $A \precsim B$ and $B \precsim A$. Now suppose π is an isomorphism of A onto a subalgebra of B. Then π need not be normal. It will only be normal if its range is a monotone closed subalgebra of B. In particular, if A is a monotone closed subalgebra of B, then by taking the natural injection as ϕ, we see that $A \precsim B$.

Lemma 2 *Let A, B, C be in Ω. If $A \precsim B$ and $B \precsim C$ then $A \precsim C$. Also $A \precsim A$.*

Proof There exists a normal, faithful positive linear map $\phi : A \to B$ and there exists a normal, faithful positive linear map $\psi : B \to C$.

Then $\psi \circ \phi : A \to C$ is a normal, faithful positive linear map.

The identity map from A to A is a surjective isomorphism and so, by the remarks above, $A \precsim A$. □

Lemma 3 *Let A and B be in Ω. Then $A \oplus B$ is also in Ω.*

Proof Since $H^\#$ is an infinite dimensional Hilbert space it is isomorphic to the direct sum of two isomorphic copies of itself, $H_1 \oplus H_2$. Then A is isomorphic to $A_1 \subset L(H_1)$ and B is isomorphic to $B_2 \subset L(H_2)$. Then $A \oplus B$ is isomorphic to $A_1 \oplus B_1$ and clearly $A_1 \oplus B_1$ can be identified with a subalgebra of $L(H_1 \oplus H_2) = L(H^\#)$. It is straightforward to verify that $A \oplus B$ is monotone complete. So $A \oplus B$ is in Ω. □

It is clear that Lemma 3 implies that Ω is closed under the taking of finite direct sums. In fact more is true:

Lemma 4 *Let (A_n) $(n = 1, 2, \dots)$ be a sequence of algebras in Ω. Then the infinite direct sum $\oplus A_n$ is in Ω.*

Proof The proof is essentially the same as in the preceding lemma, except we split $H^\#$ into a countable direct sum of isomorphic copies of itself. □

Remark 4 If, in Lemma 4, each of the algebras A_n is a small C^*-algebra then $\oplus A_n$ is also small.

Lemma 5 *Let $A_1 \precsim B_1$ and $A_2 \precsim B_2$ where A_1, B_1, A_2 and B_2 are in Ω. Then $A_1 \oplus A_2 \precsim B_1 \oplus B_2$.*

Proof By hypothesis, there exist faithful, normal positive linear maps $\phi_j : A_j \to B_j$ for $j = 1, 2$.

We define $\psi : A_1 \oplus A_2 \to B_1 \oplus B_2$ by $\psi(a_1 \oplus a_2) = \phi_1(a_1) \oplus \phi_2(a_2)$. Then it is straightforward to verify that ψ is a faithful, normal positive linear map. □

We now define a relation $\sim$ on Ω by $A \sim B$ if $A \precsim B$ and $B \precsim A$.

Lemma 6 *The relation $\sim$ is an equivalence relation on Ω.*

Proof By Lemma 2, $A \sim A$. Again by Lemma 2, if $A \sim B$ and $B \sim C$ then $A \sim C$. □

Lemma 7 *If $A_1 \sim B_1$ and $A_2 \sim B_2$ then $A_1 \oplus A_2 \sim B_1 \oplus B_2$.*

Proof This follows from Lemma 5. □

We adopt the following temporary notation. For each $A \in \Omega^{\#}$ we define $[A]$ to be the set of all B in $\Omega^{\#}$ such that $B \sim A$.

Let $W = \{[A] : A$ be a closed $*$-subalgebra of $L(H^{\#})$, and A is a monotone complete C^*-algebra.

For each $B \in \Omega$, there is an isomorphism π from B onto $A \in \Omega^{\#}$; we define $w(B)$ to be $[A]$. It is clear that w is well defined. In particular $w(A) = w(B)$ if, and only if, $A \sim B$. So, by Lemma 7, $w(A_1) = w(B_1)$ and $w(A_2) = w(B_2)$ implies $w(A_1 \oplus A_2) = w(B_1 \oplus B_2)$. It follows that we can define an operation $+$ on W by setting $w(A) + w(B) = w(A \oplus B)$. The associativity of taking direct sums immediately implies that $+$ is associative on W. So $(W, +)$ is a semigroup; we shall abuse our notation and use W for both the semigroup and the underlying set.

Proposition 5 *W is an abelian semigroup with a zero element. The zero element is $w(\mathbb{C})$, where $\mathbb{C}$ is the one dimensional algebra, the complex numbers.*

Proof Consider $w(A)$ and $w(B)$. Define $\phi : A \oplus B \to B \oplus A$ by $\phi(a \oplus b) = b \oplus a$. Then ϕ is a surjective $*$- isomorphism. Hence $w(A \oplus B) = w(B \oplus A)$. So $w(A) + w(B) = w(B) + w(A)$.

Fix A. Let $\phi : A \to A \oplus \mathbb{C}$ be defined by $\phi(a) = a \oplus 0$. Then ϕ is positive, normal and faithful. So $A \precsim A \oplus \mathbb{C}$.

Now consider $\psi : A \oplus \mathbb{C} \to A$ defined by $\psi(a \oplus \lambda) = a + \lambda 1$, where 1 is the unit element of the algebra A. Then ψ is positive and normal. Suppose that $\psi(aa^* \oplus \lambda\overline{\lambda}) = 0$. Then $aa^* + |\lambda|^2 1 = 0$. So $a = 0$ and $\lambda = 0$. i.e. ψ is faithful. So $A \oplus \mathbb{C} \precsim A$. Hence

$w(A) = w(A \oplus \mathbb{C}) = w(A) + w(\mathbb{C})$. □

We shall denote the zero element of W by 0.

Proposition 6 *Each element of W is idempotent, that is, $w(A) + w(A) = w(A)$.*

Proof Let $\phi : A \oplus A \to A$ be defined by $\phi(a \oplus b) = a + b$. Then ϕ is a faithful, normal positive linear map. So $A \oplus A \precsim A$.

Now consider $\psi : A \to A \oplus A$ defined by $\psi(a) = a \oplus a$. Then ψ is a faithful, normal positive linear map. So $A \precsim A \oplus A$. □

From Lemma 2, we see that if $A_1 \sim A_2 \precsim B_1 \sim B_2$ then $A_1 \precsim B_2$. So, without ambiguity, we may define $w(A) \leq w(B)$ if, and only if $A \precsim B$.

Lemma 8 *The relation $\leq$ is a partial ordering of the semigroup W. Then $0 \leq w(A)$ for all elements of W. Also, $w(A_1) \leq w(B_1)$ and $w(A_2) \leq w(B_2)$ implies $w(A_1) + w(A_2) \leq w(B_1) + w(B_2)$.*

Proof For any A consider the positive linear map $\phi : \mathbb{C} \to A$ defined by $\phi(\lambda) = \lambda 1$. Then ϕ is faithful and normal. So $\mathbb{C} \precsim A$ hence $0 \leq w(A)$. The second part of the lemma follows from Lemma 5. □

Corollary 3 *In the partially ordered semigroup W, $w(A) + w(B)$ is the least upper bound of $w(A)$ and $w(B)$.*

Proof Since $w(A) \leq w(A)$ and $0 \leq w(B)$, we have $w(A)+0 \leq w(A)+w(B)$. Similarly $w(B) \leq w(A)+w(B)$.

Now suppose that $w(X)$ is an upper bound for $w(A)$ and $w(B)$. Then $w(A)+w(B) \leq w(X)+w(X) = w(X)$. □

Corollary 4 *$w(A) \leq w(B)$ if and only if $w(A)+w(B) = w(B)$.*

Proof $w(A) \leq w(B)$ if and only if $w(B)$ equals the least upper bound of $w(A)$ and $w(B)$. □

Proposition 7 *Let A be an algebra in Ω. Then $w(A) = 0$ if, and only if, A is a von Neumann algebra with a faithful normal state.*

Proof By Lemma 8, for any A, we have $\mathbb{C} \precsim A$. So $A \sim \mathbb{C}$ if, and only if, $A \precsim \mathbb{C}$. But this is equivalent to the existence of a faithful normal functional $\phi : A \to \mathbb{C}$. By a well known theorem of Kadison [17], the existence of a faithful normal state on a monotone complete C^*-algebra implies that A is a von Neumann algebra. Conversely, if A is a von Neumann algebra with a faithful normal state, then $A \precsim \mathbb{C}$. □

Corollary 5 *Let A be a von Neumann algebra which is a small C^*-algebra. Then $w(A) = 0$.*

Proof When a von Neumann algebra A possesses a faithful state, then, by a well known theorem of Takesaki ([33], see page 127 and 134), the state can be split into the sum of a normal state and a completely singular state. Then this normal state is always faithful. So $w(A) = 0$. Every unital small C^*-algebra has a unital bipositive injection into $L(H)$, where H is separable. Hence any small von Neumann algebra A has a faithful state and hence a faithful normal state and hence $w(A) = 0$. □

From the above we get:

Theorem 1 *W is a partially ordered, abelian semigroup with zero. Each element of W is idempotent. The natural partial ordering induced on W by $\precsim$ coincides with the partial ordering defined by $x \leq y$ if, and only if $x+y = y$.*

A join semilattice is a partially ordered set in which each pair of elements has a least upper bound.

Corollary 6 *W is a join semilattice with a least element* 0.

Proposition 8 *Let (A_n) $(n = 1,2,\dots)$ be a sequence of algebras in Ω. Let their infinite direct sum be $\oplus A_n$. Then $w(\oplus A_n)$ is the least upper bound of the countable set $\{w(A_n) : n = 1,2,\dots\}$. In other words, in W each countable set has a supremum.*

Proof Let $w(X)$ be an upper bound for $\{w(A_n) : n = 1,2,\dots\}$. Then for each n there exists a faithful positive normal linear map ϕ_n from A_n into X. By multiplying by a suitable constant, if necessary, we can suppose $||\phi_n|| \leq 1$.

For each $\mathbf{x} \in \oplus A_n$ with $\mathbf{x} = (x_n)$ $(n = 1,2,\dots)$, let $\Phi(\mathbf{x}) = \sum_{n=1}^{\infty} 2^{-n}\phi_n(x_n)$. Then $\Phi(\mathbf{x}^*\mathbf{x}) = 0$ implies that $\phi_n(x_n^* x_n) = 0$ which implies $x_n = 0$, for all n. So Φ is faithful.

Let $(\mathbf{x}^\alpha)$ be a downward directed net with infimum 0 in $\oplus A_n$ where $\mathbf{x}^\alpha \leq 1$ for all α. Choose $\varepsilon > 0$. Then for large enough N, and all α,

$$0 \leq \Phi(\mathbf{x}^\alpha) \leq \sum_{n=1}^{N} 2^{-n}\phi_n(x_n^\alpha) + \sum_{n=N+1}^{\infty} 2^{-n}1 < \sum_{n=1}^{N} 2^{-n}\phi_n(x_n^\alpha) + \varepsilon 1.$$

Hence $\bigwedge_\alpha \Phi(\mathbf{x}^\alpha) \leq \sum_{n=1}^{N} \phi_n(x_n^{\alpha_n}) + \varepsilon$. Where $\alpha_1, \alpha_2, \ldots, \alpha_N$ are arbitrary elements of the index set for the net. Hence $\bigwedge_\alpha \Phi(\mathbf{x}^\alpha) \leq \varepsilon 1$. Since ε was arbitrary it follows that Φ is normal. So $w(\oplus A_n) \leq w(X)$. It is easy to see that $w(A_q) \leq w(\oplus A_n)$ for each q. The result follows. □

Lemma 9 *Let $\phi : A \to B$ be a positive linear map which is normal. Then there exists a projection $e \in A$ such that ϕ vanishes on $(1-e)A(1-e)$ and its restriction to eAe is faithful.*

Proof Let K be the hereditary cone $\{x \in A^+ : \phi(x) = 0\}$. Let $K_0 = \{x \in K : ||x|| < 1\}$. Then, see [23, page 11], K_0 is an upward directed set. Let p be its supremum. Since ϕ is normal, $p \in K$. Also, see [23, page 11 and page 15], K_0 is an approximate unit for the hereditary subalgebra generated by K. From this it follows that p is the unit for this subalgebra, in particular, p is a projection. Suppose that $z \in (1-p)A(1-p)$ and $\phi(zz^*) = 0$. Then $zz^* \in K$. So $zz^* = pzz^* = p(1-p)zz^* = 0$. Hence ϕ is faithful when restricted to $(1-p)A(1-p)$.

On putting $e = 1-p$, the statement of the lemma follows. □

We call e the *support projection* of ϕ.

Lemma 10 *Let e be any projection in A. Then $eAe \oplus (1-e)A(1-e) \sim A$.*

Proof If $e = 0$ or $e = 1$ then the statement is trivially true. So we suppose that neither e nor $1-e$ is zero. Let $\phi : A \to eAe \oplus (1-e)A(1-e)$ be defined by $\phi(x) = exe \oplus (1-e)x(1-e)$. Then ϕ is positive, linear and normal.

Suppose $\phi(zz^*) = 0$. Then $ezz^*e = 0$ and $(1-e)zz^*(1-e) = 0$. So $ez = 0$ and $(1-e)z = 0$. Thus $z = 0$. So ϕ is faithful.

Hence $w(A) \leq w(eAe \oplus (1-e)A(1-e))$.

The natural embedding of eAe in A is an isomorphism onto a normal subalgebra. Hence $w(eAe) \leq w(A)$. Similarly $w((1-e)A(1-e)) \leq w(A)$. Since $w(A) + w(A) = w(A)$, we have $w(eAe \oplus (1-e)A(1-e)) \leq w(A)$. The result is proved. □

The next result shows that the ordered semigroup W, has the Riesz Decomposition Property. This will then imply that, regarded as a join semilattice, it is *distributive*. This is useful because there is a well developed structure theory for distributive join semilattices which can then be applied to W.

Theorem 2 *Let a, b, c be elements of W such that $a \leq b + c$. Then $a = a_1 + a_2$ such that $0 \leq a_1 \leq b$ and $0 \leq a_2 \leq c$.*

Proof Let $a = w(A)$, $b = w(B)$, and $c = w(C)$. Then there exists a faithful normal positive linear map $\phi : A \to B \oplus C$.

For each y in the algebra B and for each z in the algebra C, let $\pi_1(y \oplus z) = y$. Then π_1 is the canonical projection from $B \oplus C$ onto the first component B. Similarly, define $\pi_2 : B \oplus C \to C$.

Let $\phi_j = \pi_j \circ \phi$ for $j = 1,2$. Then each ϕ_j is positive, linear and normal. Also $\phi(x) = \phi_1(x) \oplus \phi_2(x)$ for each $x \in A$. Suppose that $\phi_1 = 0$. Then $a \leq c$. So, on putting $a = a_2$ and $a_1 = 0$, we are done. Hence we shall now suppose that $\phi_1 \neq 0$. Then, by Lemma 9, it has a non-zero support projection e. Then $\phi_1|eAe$ is a faithful normal map into B. Also $\phi(1-e) = 0 \oplus \phi_2(1-e)$. If this vanishes then, since ϕ is faithful, $e = 1$. So ϕ_1 would be faithful on A, which implies $a \leq b$. On putting $a = a_1$ and $a_2 = 0$ we would be finished. We now suppose that $\phi(1-e) = 0 \oplus \phi_2(1-e) \neq 0$.

Since ϕ_1 vanishes on $(1-e)A(1-e)$, we have that $0 \oplus \phi_2|(1-e)A(1-e) = \phi|(1-e)A(1-e)$ which is faithful.

Hence $eAe \precsim B$ and $(1-e)A(1-e) \precsim C$. On putting $a_1 = w(eAe)$ and $a_2 = w((1-e)A(1-e))$, we find that $0 \leq a_1 \leq b$ and $0 \leq a_2 \leq c$.

By appealing to Lemma 10 we have $a_1 + a_2 = a$. $\square$

Corollary 7 *Regarded as a join semilattice, W is distributive.*

When we interpret '$+$' as the lattice operation '$\vee$' this is just a straightforward translation of the statement of the theorem.

The well established theory of distributive join semi-lattices can now be applied to W. See [9]. Since we wish to keep this paper of reasonable length we shall not pursue this here. But we remark that distributivity is a key property which, in particular, leads to an elegant representation theory akin to the Stone representation for Boolean algebras.

The classification given here maps each small von Neumann algebra to the zero of the semigroup. It could turn out that W is very small and fails to distinguish between many algebras. We shall see later that this is far from true. Even when w is restricted to special subclasses of algebras, we can show that its range in W is huge, 2^c, where $c = 2^{\aleph_0}$. In the next section we shall introduce the spectroid of an algebra and show that it is, in fact, an invariant for elements of W.

4 The spectroid and representing functions

Although our main interest is focused on monotone complete C^*- algebras, in this section we shall also use the larger class of monotone σ-complete C^*-algebras.

For any non-empty set J we let $F(J)$ be the collection of all finite subsets of J, including the empty set. In particular we note that $F(\mathbb{N})$, where $\mathbb{N}$ is the set of natural numbers, is countable.

Definition 2 A representing function for a monotone σ-complete C^*-algebra, A, is a function $f : F(\mathbb{N}) \to A^+$ such that

(i) $f(k) \geq 0$ and $f(k) \neq 0$ for all k.

(ii) f is downward directed, that is, when k,l are finite subsets of $\mathbb{N}$, then $f(k \cup l) \leq f(k)$ and $f(k \cup l) \leq f(l)$.

(iii) $\bigwedge_{k \in F(\mathbb{N})} f(k) = 0$.

Let T be a set of cardinality $2^{\aleph_0} = c$. Let $\mathbf{N} : T \to \mathscr{P}(\mathbb{N})$ be an injection and let $\mathbf{N}(t)$ be infinite for each t. We do not require that $\{\mathbf{N}(t) : t \in T\}$ contains every infinite subset of $\mathbb{N}$. We shall regard T and the function $\mathbf{N}$ as fixed until further notice.

Definition 3 Let A be a monotone σ-complete C^*-algebra and let $f : F(\mathbb{N}) \to A$ be a representing function. Then let $R_{(T,\mathbf{N})}(f)$ be the subset of T defined by

$$\left\{ t \in T : \bigwedge_{k \in F(\mathbf{N}(t))} f(k) = 0 \right\}.$$

The set $R_{(T,\mathbf{N})}(f)$ is said to be represented by f in A, modulo $(T,\mathbf{N})$.

Any subset of T which can be represented in A is said to be a representing set of A (modulo $(T,\mathbf{N})$).

Definition 4 Let A be a monotone σ-complete C^*-algebra. Then the spectroid of A (modulo $(T,\mathbf{N})$), written $\partial_{(T,\mathbf{N})}A$, is the collection of all sets which can be represented in A, modulo $(T,\mathbf{N})$, by some representing function $f : F(\mathbb{N}) \to A^+$, that is,

$$\partial_{(T,\mathbf{N})}A = \{R_{(T,\mathbf{N})}(f) : \ f \text{ is a representing function for } A\}.$$

When it is clear from the context which $(T,\mathbf{N})$ is being used, we shall sometimes write ∂A.

Let us recall that $\#S$ denotes the cardinality of a set S.

Proposition 9 *Let $(T,\mathbf{N})$ be fixed and let A be any monotone σ-complete C^*-algebra of cardinality c. Then $\partial_{(T,\mathbf{N})}(A)$ is of cardinality not exceeding c.*

Proof Each element of $\partial_{(T,\mathbf{N})}(A)$ arises from a representing function for A. But the cardinality of all functions from $F(\mathbb{N})$ into A is $\#A^{F(\mathbb{N})} = c^{\aleph_0} = c$. So $\#\partial_{(T,\mathbf{N})}(A) \leq c$. □

Corollary 8 *Let $(T,\mathbf{N})$ be fixed and let A be a small monotone complete C^*-algebra. Then $\partial_{(T,\mathbf{N})}(A)$ is of cardinality not exceeding c.*

Proof By Proposition 1, $\#A = c$. □

Lemma 11 *Let $(T,\mathbf{N})$be fixed and let S be the set of all spectroids, modulo $(T,\mathbf{N})$, of monotone σ-complete C^*-algebras of cardinality c. Then $\#\mathscr{S} \leq 2^c$.*

Proof Each subset of cardinality $\leq c$ is the range of a function from $\mathbb{R}$ into $\mathscr{P}(T)$. So $\#S \leq \#\mathscr{P}(\mathbb{R})^{\mathbb{R}} \leq \#\mathscr{P}(\mathbb{R} \times \mathbb{R}) = 2^c$. □

Suppose that A and B are monotone σ-complete C^*- algebras and $\phi : A \to B$ is a faithful positive linear map. Let us recall that ϕ is σ-*normal* if, whenever (a_n) $(n = 1, 2, \dots)$ is a monotone decreasing sequence in A with $\bigwedge_{n=1}^{\infty} a_n = 0$ then $\bigwedge_{n=1}^{\infty} \phi(a_n) = 0$.

Lemma 12 *Let A and B be monotone σ-complete C^*-algebras. Let $\phi : A \to B$ be a positive, faithful σ-normal linear map.*

Let D be a downward directed subset of A^+ which is countable. Then $\bigwedge\{d : d \in D\} = 0$ if and only if $\bigwedge\{\phi(d) : d \in D\} = 0$.

Proof By Remark 1 there exists a monotone decreasing sequence (c_n) $(n = 1, 2, \ldots)$ in D such that $d \in D$ implies $d \geq c_n$ for some n. So $\bigwedge\{\phi(d) : d \in D\} = \bigwedge_{n=1}^{\infty} \phi(c_n) = \phi(\bigwedge_{n=1}^{\infty} c_n) = \phi(\bigwedge\{d : d \in D\})$.

Since ϕ is faithful, $\phi(\bigwedge\{d : d \in D\}) = 0$ if, and only if, $\bigwedge\{d : d \in D\} = 0$. The lemma follows. □

Definition 5 Let A and B be monotone σ-complete C^*-algebras. If there exists a positive, faithful σ-normal linear map $\phi : A \to B$ we write $A \precsim_\sigma B$. Then the relation $\precsim_\sigma$ is a quasi ordering of the class of monotone σ-complete C^*-algebras.

When $A \precsim_\sigma B$ and $B \precsim_\sigma A$ we say that A and B are σ-normal equivalent and write $A \sim_\sigma B$. This is an equivalence relation on the class of monotone σ-complete C^*-algebras. Clearly, if A and B are monotone complete C^*-algebras and $A \precsim B$ then $A \precsim_\sigma B$. So if $A \sim B$ it follows that $A \sim_\sigma B$.

Proposition 10 *Let $(T, \mathbf{N})$ be fixed and let A and B be monotone σ-complete C^*-algebras. Let $A \precsim_\sigma B$. Then $\partial_{(T,\mathbf{N})}(A) \subset \partial_{(T,\mathbf{N})}(B)$.*

Proof Each element of ∂A is of the form $R_{(T,\mathbf{N})}(f)$ where f is a representing function for A. It is straightforward to verify that ϕf is a representing function for B. Since ϕ is faithful it follows from Lemma 5 that $R_{(T,\mathbf{N})}(f) = R_{(T,\mathbf{N})}(\phi f)$. Thus $\partial A \subset \partial B$. □

It is clear that the spectroid is an isomorphism invariant but from Proposition 10 it is also invariant under σ-normal equivalence.

Corollary 9 *Let $(T, \mathbf{N})$ be fixed and let A and B be monotone σ-complete C^*-algebras. Let $A \sim_\sigma B$. Then $\partial_{(T,\mathbf{N})}(A) = \partial_{(T,\mathbf{N})}(B)$.*

Corollary 10 *Let A and B be monotone complete C^*-algebras with $w(A) = w(B)$. Then $\partial_{(T,\mathbf{N})}(A) = \partial_{(T,\mathbf{N})}(B)$ for any given $(T, \mathbf{N})$.*

So the spectroid is an invariant for the semigroup W and we may talk about the spectroid of an element of the semigroup.

For the rest of this section we shall consider only monotone complete C^*-algebras, although some of the results (and proofs) are still valid for monotone σ-complete C^*-algebras. Let $\mathscr{M}$ be the class of all small monotone complete C^*-algebras. We shall use W to denote the semigroup constructed in Section 3; but we shall assume that w has been restricted to the class of all small monotone complete C^*-algebras and, from now on, use W to denote the semigroup $\{w(A) : A \in \mathscr{M}\}$. (So, in effect we are taking a sub semigroup of the one constructed in Section 3, and abusing our notation by giving it the same name).

Theorem 3 *Let $(T, \mathbf{N})$ be fixed and consider only spectroids modulo $(T, \mathbf{N})$. Let $\{A_\lambda : \lambda \in \Lambda\}$ be a collection of small monotone complete C^*-algebras such that the union of their spectroids has cardinality 2^c. Then there is a subcollection $\{A_\lambda : \lambda \in \Lambda_0\}$ where Λ_0 has cardinality 2^c and $\partial(A_\lambda) \neq \partial(A_\mu)$ whenever λ and μ are distinct elements of Λ_0.*

Proof Let us define an equivalence relation on Λ by $\lambda \approx \mu$ if, and only if, $\partial(A_\lambda) = \partial(A_\mu)$. By using the Axiom of Choice we can pick one element from each equivalence class to form Λ_0. Clearly $\partial(A_\lambda) \neq \partial(A_\mu)$ whenever $\lambda$and μ are distinct elements of Λ_0. Also $\cup\{\partial(A_\lambda) : \lambda \in \Lambda_0\}$ is equal to the union of all the spectroids of the original collection. So

$$2^c = \#\cup\{\partial(A_\lambda) : \lambda \in \Lambda_0\}. \qquad (a)$$

By Corollary 8, $\#\partial(A_\lambda) \leq c$ for each $\lambda \in \Lambda_0$. Hence, from (a), $2^c \leq c \times \#\Lambda_0$. It follows that we cannot have $\#\Lambda_0 \leq c$. So $c \times \#\Lambda_0 = \#\Lambda_0$. So $2^c \leq \#\Lambda_0$. From Lemma 11 we get $\#\Lambda_0 \leq 2^c$. So $\#\Lambda_0 = 2^c$. □

Corollary 11 *Given the hypotheses of the theorem, whenever λ and μ are distinct elements of Λ_0 then $wA_\lambda \neq wA_\mu$. So A_λ is not equivalent to A_μ. In particular, they cannot be isomorphic.*

Proof Apply Corollary 10. □

We have seen that the small monotone complete C^*-algebras can be classified by elements of W and also by their spectroids. Since w maps every small von Neumann algebra to the zero of the semigroup, this classification might be very coarse, possibly W might be too small to distinguish between more than a few classes of algebras. But we shall see in Section 8 that this is far from the truth. By applying Theorem 3 for appropriate $(T,\mathbf{N})$ we shall see that $\#W = 2^c$.

Representing sets for Boolean algebras appear in [21] and the generalisation of representing functions from the context of Boolean algebras to that of monotone complete algebras is given in [15]; the notion of spectroid appears to be new.

5 Commutative algebras: general preliminaries

We shall define a topological space to be *separable* if it has a countable dense subset. This is a much weaker condition than the existence of a countable base.

Lemma 13 *Let X be a compact Hausdorff space. Then $C(X)$ is isomorphic to a (unital) closed $*$-subalgebra of ℓ^∞ if, and only if, X is separable.*

Proof First suppose that X has a countable dense subset $\{x_n : n = 1,2,\dots\}$. Then, for each $f \in C(X)$, let Hf be the sequence $(f(x_n))$ $(n = 1,2,\dots)$. Then H is an (isometric) $*$-isomorphism of $C(X)$ into ℓ^∞.

Conversely suppose that there exists an injective (unital) $*$-homomorphism H from $C(X)$ into ℓ^∞. Then, by the Gelfand-Naimark duality between compact Hausdorff spaces and commutative unital C^*-algebras, there is a surjective continuous map from the structure space of ℓ^∞onto X. But the structure space of $\ell^\infty$is $\beta\mathbb{N}$, the Stone-Cech compactification of the natural numbers. Since $\beta\mathbb{N}$ is separable, so, also is X. □

The regular σ-completion of an arbitrary C^*-algebra was defined in [38] but for a commutative unital C^*-algebra, $C(X)$, it can be identified with $B^\infty(X)/M(X)$, (which we shall denote here by $\widehat{C(X)}$ where $B^\infty(X)$ is the algebra of all bounded Baire measurable functions on X and $M(X)$ is the ideal of all f for which $\{x \in X : f(x) \neq 0\}$ is a meagre subset of X, that is, the union of countably many nowhere dense sets. Clearly there is a natural injection of $C(X)$ into $\widehat{C(X)}$, which we shall denote by j. Since ℓ^∞ is monotone complete, it follows from [35] that if there exists an injective $*$-isomorphism h of $C(X)$ into ℓ^∞, then it extends to a $*$-homomorphism H of $\widehat{C(X)}$ into ℓ^∞. From standard properties of regular σ-completions [38], H is also an injection into ℓ^∞. So the monotone σ-complete algebra $\widehat{C(X)}$ supports a faithful state and so is monotone complete. Let $\widehat{X}$ be the Gelfand-Naimark structure space of $\widehat{C(X)}$, so that $\widehat{C(X)}$ is isomorphic to $C(\widehat{X})$. Then $\widehat{X}$ is compact and extremally disconnected; by the lemma above, it is also separable.

We recall the familiar fact that the Cantor "middle third" set is homeomorphic to $\{0,1\}^{\mathbb{N}}$, which we shall also denote by $2^{\mathbb{N}}$, and call the Cantor space. It is a compact (Hausdorff) totally disconnected space whose set of clopen subsets is countable. We shall also make use of the space $\{0,1\}^{\mathbb{R}}$, which we shall denote by $2^{\mathbb{R}}$; we shall sometimes refer to this space as the Big Cantor space. It is a moderately surprising fact that the Big Cantor space is separable. This follows from a theorem of Hewitt [16].

We shall adopt the following notation. For any unital C^*-algebra A,we denote the lattice of projections in the algebra by $Proj(A)$. Whenever T is a compact Hausdorff totally disconnected space we shall use $\mathscr{K}(T)$ for the Boolean algebra of all clopen subsets of T; then $\mathscr{K}(T)$ is isomorphic to $Proj(C(T))$.

Lemma 14 *Each closed subspace of $2^{\mathbb{R}}$ is compact, totally disconnected and has a base of c open sets. Conversely, each compact Hausdorff, totally disconnected space with a base of c open sets is homeomorphic to a closed subspace of $2^{\mathbb{R}}$.*

Proof Since $2^{\mathbb{R}}$ is totally disconnected and compact, each closed subspace S is compact. Given distinct points x,y in S, there exist disjoint clopen subsets of $2^{\mathbb{R}}$, E,F such that $x \in E$ and $y \in F$. Then $E \cap S$ and $F \cap S$ are disjoint clopen subsets of S in the relative topology. So S is totally disconnected. Since c clopen sets form a base for the topology of $2^{\mathbb{R}}$, their intersections with S form a base for the relative topology.

Now suppose that T is a compact Hausdorff, totally disconnected space with a base of c open sets. Since each clopen set is compact, it is the union of finitely many open sets from the base. So $\#\mathscr{K}(T) \leq c$.

Since, see Halmos [10], $2^{\mathbb{R}}$is the Stone structure space of the free Boolean algebra on c generators, there is a Boolean homomorphism ρ from $\mathscr{K}(2^{\mathbb{R}})$ onto $\mathscr{K}(T)$. By the Stone duality between Boolean algebras and compact Hausdorff totally disconnected spaces, there is an injective continuous map ρ^* from T into $2^{\mathbb{R}}$. Because T is compact and $2^{\mathbb{R}}$ is Hausdorff, ρ^* is a homeomorphism. □

Corollary 12 *Let T be a separable, totally disconnected compact Hausdorff space then T is homeomorphic to a closed subspace of $2^{\mathbb{R}}$.*

Proof By Lemma 13, $C(T)$ has an injection into ℓ^∞. So $\#C(T) \leq \#\ell^\infty = c$. It follows that $\#\mathscr{K}(T) = \#\mathrm{Pr}(C(T)) \leq \#C(T) \leq c$. So the clopen subsets of T form a base of at most c generators. □

Corollary 13 *Let B be a unital commutative monotone σ-complete C^*-algebra of cardinality c. Let E be its Gelfand-Naimark structure space. Then E is homeomorphic to a closed subspace of $2^{\mathbb{R}}$.*

Proof Since the algebra is monotone σ-complete, E is totally disconnected, in fact, basically disconnected. Then $\#\mathscr{K}(E) \leq \#C(E) = c$. So, by Lemma 14, E is homeomorphic to a closed subspace of the Big Cantor space. □

Corollary 14 *Let B be a small commutative monotone complete C^*-algebra. Let E be its Gelfand-Naimark structure space. Then E is homeomorphic to a closed subspace of $2^{\mathbb{R}}$.*

Proof Since the algebra is small its cardinality is c. So the result follows immediately from the preceding corollary. □

We see from the above discussion that to construct a monotone complete C^*-algebra which can be embedded as a closed $*$-subalgebra of ℓ^∞, we can proceed as follows: Take a countable subset of the Big Cantor space. Then its closure is a separable compact totally disconnected space, K, say. Then $B^\infty(K)/M(K)$, the regular σ-completion of $C(K)$, is also embeddable as a subalgebra of ℓ^∞; so it supports a faithful state and hence is monotone complete. Conversely *every* commutative monotone complete C^*-algebra which can be embedded in ℓ^∞, arises in this way. In the next section, by taking carefully chosen countable subsets of the Big Cantor space and applying the process outlined here, we will modify the approaches of Monk-Solovay [21] and of Hamana [15] to show the existence of huge numbers of commutative subalgebras which are monotone complete and mutually non-equivalent. We will also show that each of the subalgebras of ℓ^∞ we construct has a natural representation in the form $B^\infty(2^{\mathbb{N}})/J$, where J is a σ-ideal of $B^\infty(2^{\mathbb{N}})$ and J does not contain any non-zero continuous function. Before carrying out this programme we shall gather together a few more generalities.

For any topological space E we let $Top(E)$ be the collection of all open subsets of E. When Y is a subset of E we denote its closure by clE and its interior by $intE$. Then we recall that an open set U is said to be a *regular open set* if $U = int(clU)$. Let $Rg(E)$ be the collection of all regular open subsets of E.

Let X be any infinite countable subset of $2^{\mathbb{R}}$ and let its closure be K. Then K is compact Hausdorff and totally disconnected. Clearly K is separable and hence, see above, $\widehat{K}$ is also separable. Here $\widehat{K}$ is the compact extremally disconnected structure space of $B^\infty(K)/M(K)$.

By elementary Boolean algebra theory, see [10], the Dedekind completion of the Boolean algebra $\mathscr{K}(K)$ can be identified with $Rg(K)$, which, in turn, is isomorphic to the Boolean algebra of projections in $B^\infty(K)/M(K)$. Furthermore, each non empty $U \in Rg(K)$ contains a non empty clopen subset of K. It follows easily from this that $Rg(K)$ has an atom if, and only if, K has an isolated point.

Proposition 11 *Let X, K be as above. Suppose that no point of X is an isolated point of K. Then the commutative monotone complete C^*-algebra $C(\widehat{K})$ does not possess any normal states.*

Proof By the remarks above, the Boolean algebra of projections in $C(\widehat{K})$ contains an atom, if and only if, K contains an isolated point $\{y\}$.

Since X is dense in K, when $\{y\}$ is an isolated point of K then $y \in X$. So our assumption implies that $C(\widehat{K})$ is non atomic; hence $\widehat{K}$ does not have any isolated points.

By Lemma 13 the compact extremally disconnected space $\widehat{K}$ has a countable dense subset D. Because none of the points of D is isolated, D is a meagre subset of $\widehat{K}$. By an argument of Dixmier [5] this implies that there are no non-zero positive normal functionals on $C(\widehat{K})$. □

Let A be a monotone complete C^*-algebra. Let us recall that by a theorem of Kadison [17], A is a von Neumann algebra if, and only if, it has a separating family of normal states. When A does not possess any normal states it is said to be *wild*.

The following lemma is well known but as we shall make repeated use of it in the next section we include a brief argument for the convenience of the reader.

Lemma 15 *Let X, K be as above. Let $U \in TopK$ then $clU = cl(U \cap X)$. In particular, when U is a clopen set, $U = cl(U \cap X)$.*

Proof Let $y \in clU$. So given any open set V with $y \in V$ we must have $V \cap U \neq \varnothing$. Since X is dense in K, $(V \cap U) \cap X \neq \varnothing$. So $V \cap (U \cap X) \neq \varnothing$. Thus $y \in cl(U \cap X)$. The reverse inclusion is trivial. □

We shall need the following corollary in the next section.

Corollary 15 *Let $L \cap X = M \cap F \cap X$, where L, M are clopen subsets of K and F is a closed subset of K. Then $L = M \cap intF$.*

Proof Applying the preceding lemma, $L = cl(L \cap X) \subset cl(F \cap X) \subset F$. So $L \subset intF$.

Again, by the lemma, $L \cap X \subset M \cap X$ implies $L \subset M$. Thus $L \subset M \cap intF$.

But $(M \cap intF) \cap X \subset M \cap F \cap X = L \cap X$. By applying the lemma again, $cl(M \cap intF) \subset L$. Hence $L = M \cap intF$. □

6 Constructing commutative examples

As before, we use the notation $F(S)$ to denote the collection of all finite subsets (including the empty set) of a set S.

Constructing commutative monotone complete C^*-algebras is equivalent to constructing complete Boolean algebras. It was not clear from the original constructions of Monk-Solovay [21] that their examples could support free ergodic group actions, a very important point for the non-commutative theory. Hamana [15] produced elegant constructions which did support Z-actions. In Sections 6 and 7 we have modified his approach by using the Big Cantor Space. He starts with a countable set and equips it with an exotic topology (he also deals with higher cardinalities, but we stick to what is relevant to this paper). Instead we start with a countable subset of the Big Cantor Space and take its closure. It is then immediate that this closure is a compact Hausdorff space, totally disconnected and with a countable dense set. Our constructions in Sections 6 and 7 make fundamental use of those of Hamana, our indebtedness to him is obvious, but we believe our approach is slightly more transparent and easier for the reader. We are also able to show that the algebras we construct are quotients of the bounded Borel functions on $2^{\mathbb{N}}$ by σ-ideals. We can find 2^c inequivalent algebras, such that *every* infinite countable group G has a free ergodic action, on *each* algebra, by an action which permutes the indexes of $2^{\mathbb{N}}$, see Theorem 5.

Definition 6 A pair $(T, \mathbf{O})$ is said to be feasible if it satisfies the following conditions:

(i) T is a set of cardinality $c = 2^{\aleph_0}$; $\mathbf{O} = (O_n)$ $(n = 1, 2, \ldots)$ is an infinite sequence of non-empty subsets of T, with $O_m \neq O_n$ whenever $m \neq n$.

(ii) Let M be a finite subset of T and $t \in T \backslash M$. For each natural number m there exists $n > m$ such that $t \in O_n$ and $O_n \cap M = \varnothing$.

In other words $\{n \in \mathbb{N} : t \in O_n$ and $O_n \cap M = \varnothing\}$ is an infinite set.

An example satisfying these conditions can be obtained by putting $T = 2^{\mathbb{N}}$, the Cantor space and letting $\mathbf{O}$ be an enumeration (without repetitions) of the (countable) collection of all non-empty clopen subsets.

For the rest of this section $(T, \mathbf{O})$ will be a fixed but arbitrary feasible pair.

Definition 7 Let $(T, \mathbf{O})$ be a feasible pair and let R be a subset of T. Then R is said to be admissible if

(i) R is a subset of T, with $\#R = \#(T \backslash R) = c$.

(ii) O_n is not a subset of R for any natural number n.

Return to the example where T is the Cantor space and $\mathbf{O}$ an enumeration of the non-empty clopen subsets. Then, whenever $R \subset 2^{\mathbb{N}}$ is nowhere dense and of cardinality c, R is admissible.

Lemma 16 *Let $(T, \mathbf{O})$ be any feasible pair and let R be an admissible subset of T. Then there are 2^c subsets of R which are admissible.*

Proof Let $S \subset R$ where $\#S = c$. Then S is admissible. □

Throughout this section the feasible pair is kept fixed and the existence of at least one admissible set is assumed. For the moment, R is a fixed admissible subset of T. Later on we shall vary R.

Since $F(\mathbb{N}) \times F(T)$ has cardinality c, we can identify the Big Cantor space with $2^{F(\mathbb{N}) \times F(T)}$. For each $k \in F(\mathbb{N})$, let $f_k \in 2^{F(\mathbb{N}) \times F(T)}$ be the characteristic function of the set

$$\{(l, L) : L \in F(T \backslash R),\ l \subset k \text{ and } O_n \cap L = \varnothing \text{ whenever } n \in k \text{ and } n \notin l\}.$$

Let X_R be the countable set $\{f_k : k \in F(\mathbb{N})\}$. Let K_R be the closure of X_R in the Big Cantor space. Then K_R is a (separable) compact Hausdorff totally disconnected space with respect to the relative topology induced by the product topology of the Big Cantor space. Let $A_R = B^\infty(K_R)/M(K_R)$, the regular σ- completion of $C(K_R)$. By the discussion in the previous section, A_R is monotone complete and is a (unital) C^*-subalgebra of ℓ^∞. Furthermore, the only way it could fail to be wild, is if one of the points in X_R were an isolated point in K_R. We shall show that this does not happen; so that the algebra must be wild.

The projections in A_R form a complete Boolean algebra which satisfies the countable chain condition (because it embeds in ℓ^∞ which supports a faithful state) and which is Boolean isomorphic to $Rg(K_R)$, the Boolean algebra of regular open subsets of K_R.

For each $(k, K) \in F(\mathbb{N}) \times F(T)$ let $E_{(k,K)} = \{x \in K_R : x(k, K) = 1\}$. The definition of the product topology of the Big Cantor space implies that $E_{(k,K)}$ and its compliment $E^c_{(k,K)}$ are clopen subsets of K_R. It also follows from the definition of the product topology that finite intersections of such clopen sets form a base for the topology of K_R. Hence their intersections with X_R give a base for the relative topology of X_R. We shall see that, in fact, $\{E_{(k,K)} \cap X_R : k \in F(\mathbb{N}), K \in F(T\backslash R)\}$ is a base for the topology of X_R. To establish this we first need to prove some preliminary technical results. We shall then show that $Rg(K_R)$ is generated as a Boolean σ-algebra by the countable set $\{E_{(\{n\},\varnothing)} : n \in \mathbb{N}\}$. Using this we shall find a natural representation of A_R in the form $B^\infty(2^{\mathbb{N}})/J_R$ where J_R is a σ-ideal of the bounded Baire measurable functions on the Cantor space with $C(2^{\mathbb{N}}) \cap J_R = \{0\}$.

Lemma 17 *$E_{(k,K)} = \varnothing$ unless $K \subset T\backslash R$.*

Proof Suppose K is not a subset of $T\backslash R$. Then, for any $h \in F(\mathbb{N})$, it follows from the definition of f_h, that $f_h(k,K) = 0$. So $X_R \subset E^c_{(k,K)}$. Then by Lemma 15, $K_R = clX_R = E^c_{(k,K)}$. Thus $E_{(k,K)} = \varnothing$. □

Lemma 18 *Let $x \in X_R$ and $(k,K) \in F(\mathbb{N}) \times F(T)$. Let $x \in E^c_{(k,K)}$. Then there exists $(l,L) \in F(\mathbb{N}) \times F(T\backslash R)$ such that $x \in E_{(l,L)} \subset E^c_{(k,K)}$.*

Proof First suppose that $K \cap R \neq \varnothing$. Then by the preceding lemma $K_R = E^c_{(k,K)}$.

For any $h \in F(\mathbb{N})$, $f_h \in E_{(h,\varnothing)}$. But $E_{(h,\varnothing)} \subset K_R = E^c_{(k,K)}$. So we may now assume that $K \cap R = \varnothing$.

Let $x = f_h$. Then $f_h \in E^c_{(k,K)} \Longleftrightarrow not(k \subset h \,\&(\forall n(n \in h\backslash k \Longrightarrow O_n \cap K = \varnothing))$ $\Leftrightarrow k\backslash h \neq \varnothing$ or $\exists n_1 \in h\backslash k$ such that $O_{n_1} \cap K \neq \varnothing$.

(1) First let us deal with the situation where $k\backslash h \neq \varnothing$. Then there exists $n_0 \in k\backslash h$. Since R is admissible, we can find $t_0 \in T\backslash R$ such that $t_0 \in O_{n_0}$. Then it is straight forward to verify that $f_h \in E_{(h,\{t_0\})}$. It remains to show that $E_{(h,\{t_0\})} \cap E_{(k,K)} = \varnothing$. Suppose that this is false. Then we can find $f_g \in E_{(h,\{t_0\})} \cap E_{(k,K)}$. So $h \subset g$ and $k \subset g$. Thus $n_0 \in g\backslash h$. Then $f_g \in E_{(h,\{t_0\})}$ implies $t_0 \notin O_{n_0}$. But this is a contradiction. So

$$f_h \in E_{(h,\{t_0\})} \subset E^c_{(k,K)}.$$

(2) Now consider the case where $\exists n_1 \in h\backslash k$ such that $O_{n_1} \cap K \neq \varnothing$. Consider $E_{(\{n_1\},\varnothing)}$. It is clear that f_h is an element of this set. We now wish to show that $E_{(\{n_1\},\varnothing)} \cap E_{(k,K)} = \varnothing$. Suppose this is false. Then we can find $f_g \in E_{(\{n_1\},\varnothing)} \cap E_{(k,K)}$. Then $n_1 \in g\backslash k$. So $O_{n_1} \cap K = \varnothing$. This is a contradiction. So $f_h \in E_{(\{n_1\},\varnothing)} \subset E^c_{(k,K)}$. □

Lemma 19 *Let (l,L) and (k,K) be any elements of $F(\mathbb{N}) \times F(T\backslash R)$ such that $E_{(l,L)} \cap E_{(k,K)} \neq \varnothing$. Then*

$$E_{(l,L)} \cap E_{(k,K)} = E_{(l\cup k, L\cup K)}.$$

Proof Since $E_{(l,L)} \cap E_{(k,K)}$ is not empty and X_R is dense in K_R, $E_{(l,L)} \cap E_{(k,K)} \cap X_R$ is not empty. Let $f_h \in E_{(l,L)} \cap E_{(k,K)} \cap X_R$. Then $l \subset h$ and $k \subset h$. So $l \cup k \subset h$. Also

$$O_n \cap L = \varnothing \text{ for all } n \in h\backslash l \text{ and } O_n \cap K = \varnothing \text{ for all } n \in h\backslash k. \qquad (\#)$$

Since $k\backslash l \subset h\backslash l$ and $l\backslash k \subset h\backslash k$ we have

$$O_n \cap L = \varnothing \text{ for all } n \in (l\cup k)\backslash l \text{ and } O_n \cap K = \varnothing \text{ for all } n \in (l\cup k)\backslash k. \qquad (\#\#)$$

From (#) we have $O_n \cap (L \cup K) = \varnothing$ whenever $n \in h \backslash (l \cup k)$. So $f_h \in E_{(l \cup k, L \cup K)}$. Thus $(E_{(l,L)} \cap E_{(k,K)} \cap X_R) \subset E_{(l \cup k, L \cup K)} \cap X_R$. Hence $E_{(l,L)} \cap E_{(k,K)} \subset E_{(l \cup k, L \cup K)}$.

By the above, $E_{(l \cup k, L \cup K)}$ is not empty. So $E_{(l \cup k, L \cup K)} \cap X_R$ is not empty. Let $f_g \in E_{(l \cup k, L \cup K)}$. Then $l \cup k \subset g$. Also, for all $n \in g \backslash (l \cup k)$, we have $O_n \cap (L \cup K) = \varnothing$. By (##) we also have $O_n \cap L = \varnothing$ for $n \in (l \cup k) \backslash l$. Hence $f_g \in E_{(l,L)}$. Similarly $f_g \in E_{(k,K)}$.

It now follows that $E_{(l \cup k, L \cup K)} \cap X_R$ is a subset of $E_{(l,L)} \cap E_{(k,K)}$. Taking closures and applying Lemma 15, gives $E_{(l,L)} \cap E_{(k,K)} = E_{(l \cup k, L \cup K)}$. □

Lemma 20 *Let U be an open subset of K_R and $x \in U \cap X_R$. Then there exists $(k,K) \in F(\mathbb{N}) \times F(T \backslash R)$ such that*

$$x \in E_{(k,K)} \subset U.$$

Proof It follows from the definition of the product topology on the Big Cantor space and Lemma 18, that $x \in \cap_{j=1}^{q} E_{(h(j),H(j))} \subset U$ where $(h(j),H(j)) \in F(\mathbb{N}) \times F(T \backslash R)$ for $j = 1,2,\ldots,q$.

Let $k = \cup_{j=1}^{q} h(j)$ and let $K = \cup_{j=1}^{q} H(j)$. Then, by repeated applications of Lem-ma 19, we have $x \in E_{(k,K)} \subset U$. □

Corollary 16 *Let U be a non-empty regular open subset of K_R. Then there exists a sequence $(k(j),K(j))$ $(j = 1,2,\ldots)$ in $F(\mathbb{N}) \times F(T \backslash R)$ such that, in the complete Boolean algebra of regular open subsets of K_R,*

$$U = \bigvee_{j=1}^{\infty} E_{(k(j),K(j))}.$$

Proof Since $X_R \cap U$ is a countable set it can be enumerated by (x_j) $(j = 1,2,\ldots)$. By Lemma 20, for each j, we can find $(k(j),K(j))$ in $F(\mathbb{N}) \times F(T \backslash R)$ such that $x_j \in E_{(k(j),K(j))} \subset U$. So $X_R \cap U \subset \cup_{j=1}^{\infty} E_{(k(j),K(j))} \subset U$. On taking closures, we get $clU \subset cl \cup_{j=1}^{\infty} E_{(k(j),K(j))} \subset clU$.

Because U is a regular open set, $U = int(clU)$. We recall that the supremum. of a sequence of regular open sets in $Rg(K_R)$ is formed by taking the closure of their union, and then taking the interior of that set. So $U = \bigvee_{j=1}^{\infty} E_{(k(j),K(j))}$. □

The following technical lemma will not be needed until the next section but it seems natural to prove it in this section.

Lemma 21 *Let (l,L) and (k,K) be in $F(\mathbb{N}) \times F(T \backslash R)$. Let $E_{(l,L)} = E_{(k,K)}$. Then $l = k$ and $L = K$.*

Proof For $h \in F(\mathbb{N})$, $f_h \in E_{(l,L)} \Longleftrightarrow l \subset h$, and whenever $n \in h \backslash l$ then $O_n \cap L = \varnothing$.

It follows that $f_l \in E_{(l,L)}$ and so $f_l \in E_{(k,K)}$. Thus $k \subset l$. Similarly we can show that $l \subset k$. So $l = k$.

Suppose that L is not a subset of K. Then there exists $t \in L \backslash K$. Then, by feasibility, there exists m such that $m \notin k$, $t \in O_m$ and $O_m \cap K = \varnothing$. Let $h = k \cup \{m\}$. Hence $f_h \in E_{(k,K)}$. So then $f_h \in E_{(k,L)}$. So $O_m \cap L = \varnothing$. But $t \in O_m \cap L$. This contradiction shows that $L \subset K$.

Similarly $K \subset L$. □

Corollary 17 *Let (l,L) and (k,K) be in $F(N)\times F(T\backslash R)$. If $E_{(l,L)}\subset E_{(k,K)}$ then $k\subset l$ and $K\subset L$.*

Conversely, if $k\subset l$ and $K\subset L$ then either $E_{(l,L)}\cap E_{(k,K)}=\varnothing$ or $E_{(l,L)}\subset E_{(k,K)}$.

Proof First suppose $E_{(l,L)}\subset E_{(k,K)}$. Then $E_{(l,L)}=E_{(l,L)}\cap E_{(k,K)}$. Since $f_l(l,L)=1$ this intersection is not empty. So, by Lemma 19, $E_{(l,L)}=E_{(l\cup k,L\cup K)}$. By Lemma 21, $l=l\cup k$ and $L=L\cup K$. i.e. $k\subset l$ and $K\subset L$.

Conversely, let $k\subset l$ and $K\subset L$. By Lemma 19, either $E_{(l,L)}\cap E_{(k,K)}=\varnothing$ or $E_{(l,L)}\cap E_{(k,K)}=E_{(l\cup k,L\cup K)}=E_{(l,L)}$. □

Proposition 12 *$A_R=B^\infty(K_R)/M(K_R)=C(\widehat{K_R})$ is wild and non-atomic.*

Proof It follows from the work of the preceding Section that it suffices to show that none of the elements of X_R is an isolated point in K_R.

Suppose this is false and f_g is an isolated point. Then, by Lemma 20, for some $k\in F(\mathbb{N})$ and $K\in F(T\backslash R)$, $E_{(k,K)}=\{f_g\}$.

Since K is a finite set and $T\backslash R$ is infinite, we can find $t_0\in(T\backslash R)\backslash K$. It now follows from the definition of feasibility that we can find $n_0\notin g$ such that $t_0\in O_{n_0}$ and $O_{n_0}\cap K=\varnothing$. Let $h=g\cup\{n_0\}$. Then $k\subset h$ and, for $n\in h\backslash k, O_n\cap K=\varnothing$. So $f_h\in E_{(k,K)}$. Hence $f_h=f_g$. But $f_g(\{n_0\}\cup k,K)=0$ whereas $f_h(\{n_0\}\cup k,K)=1$. This is a contradiction. So the proposition is proved. □

Lemma 22 *For each $k\in F(\mathbb{N})$ and $t\in T\backslash R$ we have*

$$E_{(k,\{t\})}=\bigcap_{n\in k}E_{(\{n\},\varnothing)}\cap int\left(\bigcap_{n\notin k,t\in O_n}(K_R\backslash E_{(\{n\},\varnothing)})\right).$$

Proof By the last corollary in preceding section, it suffices to prove that

$$X_R\cap E_{(k,\{t\})}=X_R\cap\bigcap_{n\in k}E_{(\{n\},\varnothing)}\cap\bigcap_{n\notin k,t\in O_n}(K_R\backslash E_{(\{n\},\varnothing)}).\qquad(\#)$$

Let $f_g\in X_R\cap E_{(k,\{t\})}$. So $f_g(k,\{t\})=1$. Thus

$$k\subset g\qquad(a)$$

and

$$\text{for every } n\in g\backslash k \text{ we have } t\notin O_n.\qquad(b)$$

So, by (a), $f_g(\{n\},\varnothing)=1$ for each $n\in k$. Thus $f_g\in E_{(\{n\},\varnothing)}$ for every $n\in k$.

Now consider $n\notin k$. If $n\in g$ then $n\in g\backslash k$. So $O_n\cap\{t\}=\varnothing$. Hence if $n\notin k$ and $t\in O_n$ then $n\notin g$. So $f_g(\{n\},\varnothing)=0$. Thus $f_g\in K_R\backslash E_{(\{n\},\varnothing)}$. It now follows that f_g is an element of the right hand side of (#).

Conversely, let us take f_h to be an element of the right hand side of (#). Then $f_h(\{n\},\varnothing)=1$ for each $n\in k$. So $k\subset h$. Now fix $n\in h\backslash k$.

Then $f_h(\{n\},\varnothing)=1$. If $t\in O_n$ then $f_h\in(X_R\backslash E_{(\{n\},\varnothing)})$ which would imply $f_h(\{n\},\varnothing)=0$. Hence $t\notin O_n$. It follows that $f_h(k,\{t\})=1$. So the equality (#) is established. □

Proposition 13 *$Rg(K_R)$, the complete Boolean algebra of regular open subsets of K_R, is the smallest σ-complete subalgebra of itself which contains the countable set $\{E_{(\{n\},\varnothing)} : n = 1,2,\dots\}$.*

Proof Let $\mathscr{S}$ be the σ-subalgebra of $Rg(K_R)$ generated by $\{E_{(\{n\},\varnothing)} : n = 1,2,\dots\}$. Fix $k \in F(\mathbb{N})\backslash\{\varnothing\}$ and consider $E_{(k,\varnothing)}$. Then $f_k \in E_{(\{n\},\varnothing)}$ for each $n \in k$. So, by Lemma 19, $E_{(k,\varnothing)} = \cap_{n\in k} E_{(\{n\},\varnothing)}$. Hence $E_{(k,\varnothing)} \in \mathscr{S}$. We observe that $E_{(\varnothing,\varnothing)} \cap X_R = \{f_g : g \in F(\mathbb{N})\} = X_R$. So $E_{(\varnothing,\varnothing)} = K_R$.

We now consider $E_{(k,K)}$ where $K \neq \varnothing$. If $E_{(k,K)} \neq \varnothing$ then $K \subset T\backslash R$. So $K = \{t_1,t_2,\dots,t_n\}$ where each t_j is in $T\backslash R$.

By Lemma 22, $E_{(k,\{t\})} \in \mathscr{S}$ for each $k \in F(\mathbb{N})$ and $t \in T\backslash R$. Also $f_k \in E_{(k,\{t\})}$. It now follows from Lemma 19 that $E_{(k,K)} \in \mathscr{S}$. We can now apply Corollary 16 to deduce that $Rg(K_R) \subset \mathscr{S}$. $\square$

Corollary 18 *Let q_R be the canonical quotient homomorphism of $B^\infty(K_R)$ onto $B^\infty(K_R)/M(K_R)$. Let B be a Boolean σ-subalgebra of the Baire subsets of K_R such that $E_{(\{n\},\varnothing)} \in B$ for each $n \in \mathbb{N}$. Then $\{q_R(\chi_S) : S \in B\}$ is the set of all projections in $B^\infty(K_R)/M(K_R)$.*

Proof This follows from Proposition 13 and the observation that the map $O \longrightarrow q_R(\chi_O)$ is a Boolean isomorphism from $Rg(K_R)$ onto the Boolean algebra of all projections in $B^\infty(K_R)/M(K_R)$. $\square$

Definition 8 Let

$$\mathbf{N} : T \longmapsto \mathscr{P}(\mathbb{N})$$

be the map defined by

$$\mathbf{N}(t) = \{n \in \mathbb{N} : t \in O_n\}$$

for each $t \in T$.

We remark that the definition of feasibility implies that $\mathbf{N}(t)$ is an infinite set for every $t \in T$. Feasibility also implies that it is an injective map. Its definition is independent of any choice of R.

Proposition 14 *For each $t \in T$ let $C(t)$ be the closed set defined by*

$$C(t) = \bigcap_{n\in\mathbf{N}(t)} (K_R\backslash E_{(\{n\},\varnothing)}).$$

Then $C(t)$ has empty interior if, and only if, $t \in R$.

Proof We recall that $f_g(\{n\},\varnothing) = 0$ if, and only if, $n \notin g$. So $f_g \in C(t)$ if, and only if, $g \cap \mathbf{N}(t) = \varnothing$.

First assume that $t \in T\backslash R$. Then $f_h \in E_{(\varnothing,\{t\})} \iff (n \in h \text{ implies } t \notin O_n) \iff h \cap \mathbf{N}(t) = \varnothing \iff f_h \in C(t)$. Thus $f_\varnothing \in E_{(\varnothing,\{t\})} \cap X_R = C(t) \cap X_R$. So $C(t)$ is non-empty and, by applying Lemma 15, is equal to the clopen set $E_{(\varnothing,\{t\})}$.

Conversely, let us assume that $C(t)$ has non-empty interior. So there exists $(k,K) \in F(\mathbb{N}) \times F(T\backslash R)$ such that $\varnothing \neq E_{(k,K)} \subset C(t)$.

First suppose that $t \notin K$. Since $(T,\mathbf{O})$ is feasible we can find $n \notin k$ such that $t \in O_n$ and $O_n \cap K = \varnothing$. Let $h = \{n\} \cup k$. Then it follows from this that $f_h \in E_{(k,K)}$.

So $f_h \in C(t)$. Thus $h \cap \mathbf{N}(t) = \varnothing$. In particular, $t \notin O_n$. This contradiction shows that we must have $t \in K \subset T \backslash R$.

So $C(t)$ has non-empty interior if, and only if, $t \in T \backslash R$. The required result follows. □

We shall now see how to represent $A_R = B^{\infty}(K_R)/M(K_R) = C(\widehat{X})$ as a quotient of the algebra of Baire functions on the (classical) Cantor space. The key fact which makes this possible is the existence of a countable set of generators.

Let Γ be a map from the Big Cantor space onto the small Cantor space, defined as follows.

For $x \in 2^{F(\mathbb{N}) \times F(T)}$ let $\Gamma(x)(n) = x((\{n\}, \varnothing))$ for $n = 1, 2, \ldots$. Then Γ is a map from the Big Cantor space into $2^{\mathbb{N}}$, the classical Cantor space.

Let $J = \{(\{n\}, \varnothing) : n \in \mathbb{N}\}$. Then, trivially, we may identify $2^{\mathbb{N}}$ with 2^J. So Γ may be regarded as a restriction map and, by the definition of the topology for product spaces, it is continuous.

Let $\Sigma = \{y \in 2^{\mathbb{N}} : y(n) = 0$ for all but finitely many $n\}$. Then Σ is a countable dense subset of $2^{\mathbb{N}}$ such that $\Gamma[X_R] = \Sigma$. Let Γ_R be the restriction of Γ to K_R. Then $\Gamma_R$is a continuous map from the compact Hausdorff space K_R onto a compact Hausdorff space. Since X_R is dense in K_R, it follows that $\Gamma_R[K_R] = 2^{\mathbb{N}}$. This map $\Gamma_R$is never an open map, see the remarks at the end of this section.

Let $E_n = \{y \in 2^{\mathbb{N}} : y(n) = 1\}$. Then $\Gamma_R^{-1}[E_n] = E_{(\{n\}, \varnothing)}$ for $n = 1, 2, \ldots$.

Since Γ_R is continuous, it follows that whenever $f \in B^{\infty}(2^{\mathbb{N}})$ then $f \circ \Gamma_R$ is in $B^{\infty}(K_R)$. We define a σ-homomorphism γ_R from $B^{\infty}(2^{\mathbb{N}})$ to $B^{\infty}(K_R)$ by $\gamma_R(f) = f \circ \Gamma_R$. As in Corollary 18, we let q_R be the canonical quotient homomorphism of $B^{\infty}(K_R)$ onto $B^{\infty}(K_R)/M(K_R)$.

Definition 9 Let $I_R = \{f \in B^{\infty}(2^{\mathbb{N}}) : f \circ \Gamma_R \in M(K_R)\}$.

Theorem 4 *I_R is a σ-ideal of $B^{\infty}(2^{\mathbb{N}})$ and $B^{\infty}(2^{\mathbb{N}})/I_R$ is isomorphic to $B^{\infty}(K_R)/M(K_R) \approx C(\widehat{K_R}) = A_R$. Also $I_R \cap C(2^{\mathbb{N}}) = \{0\}$.*

Proof The mapping $q_R \circ \gamma_R$ is a σ-homomorphism, so its kernel is a σ-ideal. But

$$q_R \circ \gamma_R(f) = 0 \Longleftrightarrow \gamma_R(f) \in M(K_R) \Longleftrightarrow f \circ \Gamma_R \in M(K_R) \Longleftrightarrow f \in I_R.$$

Thus I_R is a σ-ideal and $B^{\infty}(2^{\mathbb{N}})/I_R$ is isomorphic to $q_R \circ \gamma_R[B^{\infty}(2^{\mathbb{N}})] \subset B^{\infty}(K_R)/M(K_R)$.

We observe that γ_R maps the characteristic function of E_n to the characteristic function of $E_{(\{n\}, \varnothing)}$. It now follows from Corollary 18 that the range of $q_R \circ \gamma_R$ contains every projection in $B^{\infty}(K_R)/M(K_R)$. Since the range of $q_R \circ \gamma_R$ is a closed subalgebra of $B^{\infty}(K_R)/M(K_R)$ it must coincide with $B^{\infty}(K_R)/M(K_R)$.

Finally, if $f \in I_R \cap C(2^{\mathbb{N}})$, then $f \circ \Gamma_R$ is a continuous function in $M(K_R)$. So $\{y \in K_R : f \circ \Gamma_R(y) \neq 0\}$ is an open meagre set. So, by the Baire Category Theorem for compact Hausdorff spaces, this set is empty i.e. $f = 0$. This completes the proof. □

We denote the natural quotient homomorphism from $B^{\infty}(2^{\mathbb{N}})$ to $B^{\infty}(2^{\mathbb{N}})/I_R$ by π_R.

To avoid "subscripts of subscripts" we shall denote the characteristic function of a set S by $\chi(S)$.

We now let $D_n = 2^{\mathbb{N}} \backslash E_n$. So γ_R maps $\chi(D_n)$ to $\chi(E^c_{(\{n\},\varnothing)})$. For each $k \in F(\mathbb{N})\backslash\{\varnothing\}$ let $d_k = \chi(\cap_{n\in k} E^c_{(\{n\},\varnothing)})$. Then γ_R maps $\chi(\cap_{n\in k} D_n)$ to d_k.

Because γ_R is a σ-homomorphism, it maps $\chi(\cap_{n\in \mathbf{N}(t)} D_n)$ to $\chi(C(t))$. Then, by Proposition 14, $\chi(C(t)) \in M(K_R)$ if and only if $t \in R$. So $\chi(\cap_{n\in \mathbf{N}(t)} D_n) \in I_R$ if and only if $t \in R$.

Proposition 15 *When $R \neq S$ then $I_R \neq I_S$.*

Proof Without loss of generality, we may suppose that there exists $t \in R\backslash S$. Then $\chi(\cap_{n\in \mathbf{N}(t)} D_n) \in I_R$ but $\chi(\cap_{n\in \mathbf{N}(t)} D_n) \notin I_S$. So $I_R \neq I_S$. □

Corollary 19 *There are 2^c distinct ideals I_R.*

Proof By Lemma 16 there are 2^c admissible sets. □

Remark 5 When $R \neq S$ then $I_R \neq I_S$. But it does not necessarily follow that $B^\infty(2^{\mathbb{N}})/I_R$ is not isomorphic to $B^\infty(2^{\mathbb{N}})/I_S$. To show that there are 2^c algebras A_R which are not equivalent and hence, in particular, not isomorphic, we make use of the machinery of representing functions and spectroids, modulo $(T, \mathbf{N})$, where $\mathbf{N}$ is the map defined in Definition 8.

We define a particular representing function for $B^\infty(2^{\mathbb{N}})/I_R$ by defining $f_R(k) = \pi_R(\chi(\cap_{n\in k} D_n))$ for $k \neq \varnothing$, and putting $f_R(\varnothing) = 1$.

Then we have:

Lemma 23 *For each admissible R the function f_R is a representing function for $B^\infty(2^{\mathbb{N}})/I_R$. Then R is represented by f_R, modulo $(T, \mathbf{N})$. In other words, $R \in \partial_{(T,\mathbf{N})}(B^\infty(2^{\mathbb{N}})/I_R)$.*

Proof First we note that $\cap_{n\in k} D_n$ is a non-empty clopen set for each finite set k. So $\chi(\cap_{n\in k} D_n)$ is a non-zero continuous function. Hence it is not in I_R. It is clear that f_R is downward directed. Now consider $\bigwedge_{k\in F(\mathbb{N})} f_R(k) = \pi_R(\chi(\cap_{n=1}^\infty D_n))$. But $\cap_{n=1}^\infty D_n$ is a single point set. So $\pi_R(\chi(\cap_{n=1}^\infty D_n))$ is zero or an atomic projection. Since $B^\infty(2^{\mathbb{N}})/I_R \approx A_R$ which has no atoms, $\bigwedge_{k\in F(\mathbb{N})} f_R(k) = 0$. Thus f_R is a representing function.

We now calculate $R_{(T,\mathbf{N})}(f_R) = \{t \in T : \bigwedge_{k\in F(\mathbb{N}(t))} f_R(k) = 0\}$. We have $\bigwedge_{k\in F(\mathbb{N}(t))} f_R(k) = 0$ precisely when $\chi(\cap_{n\in \mathbf{N}(t)} D_n) \in I_R$.

In the notation of Proposition 14, $\cap_{n\in \mathbf{N}(t)} D_n = C(t)$. So, by applying Proposition 14, we see that $\chi(C(t)) \in I_R$ precisely when $t \in R$. Thus $R = R_{(T,\mathbf{N})}(f_R)$.

So R is in $\partial_{(T,\mathbf{N})}(B^\infty(2^{\mathbb{N}})/I_R) = \partial_{(T,\mathbf{N})}(A_R)$. □

Corollary 20 *Let $\mathscr{R}$ be the collection of all admissible subsets of T. Then there exists $\mathscr{R}_0 \subset \mathscr{R}$, with $\#\mathscr{R}_0 = 2^c$ and such that $\{A_R : R \in \mathscr{R}_0\}$ is a set for which A_R is not equivalent to A_S unless $R = S$. Also the spectroid of A_R, modulo $(T, \mathbf{N})$, is not equal to the spectroid of A_S, modulo $(T, \mathbf{N})$, when $R \neq S$.*

Proof By Lemma 16 $\#\mathscr{R} = 2^c$. From Lemma 23, for each $R \in \mathscr{R}, R \in \partial_{(T,\mathbf{N})}(A_R)$. The result now follows from Theorem 3 and Corollary 10. □

Remark 6 We can prove that none of the algebras constructed above is isomorphic to the Dixmier algebra. If, for a given R, the continuous map Γ_R were open then it is easy to see that $B^\infty(2^{\mathbb{N}})/I_R$ would be isomorphic to the Dixmier algebra. So Γ_R is never open.

In the construction of commutative monotone complete algebras described in this section we have assumed that $\#(T\backslash R)$, the cardinality of the compliment of the admissible set used, is 2^c. If we replaced this assumption by requiring $\#(T\backslash R) = \aleph_0$, then the constructions would still work but we can show that the algebras obtained would all be isomorphic to the Dixmier algebra.

7 Group actions

One of the motives for studying group actions on commutative monotone complete C^*-algebras, is that, when the actions are free and ergodic, they give rise to constructions of monotone complete factors [14], [15], [25], [31], [32]. Each of these factors contains a maximal abelian subalgebra which is isomorphic to the initial commutative algebra. The spectroid of such a factor contains the spectroid of the commutative subalgebra.

Let X be any compact Hausdorff space. Let G be a countably infinite discrete group. Let $g \to \alpha_g$ be an action of G on X as homeomorphisms. (Equivalently $g \to \alpha_{g^{-1}}$ is an anti-action). Then $f \to f \circ \alpha_{g^{-1}}$ is an action of G on $C(X)$.

For any Borel (Baire) function f on X, $f \circ \alpha_g$ is another Borel (Baire) function on X. Since homeomorphisms map meagre sets to meagre sets, $f \in M(X)$ if and only if $f \circ \alpha_g \in M(X)$. Let q be the canonical quotient homomorphism of $B^\infty(X)$ onto $B^\infty(X)/M(X)$. Then, for each $g \in G$ and each $f \in B^\infty(X)$, we can define $\overline{\alpha}_g(q(f)) = q(f \circ \alpha_{g^{-1}})$. So the action α induces an action $\overline{\alpha}$, as $*$-automorphisms, on $B^\infty(X)/M(X)$.

The following lemma is elementary but, since we shall make essential use of it, we give a proof.

Lemma 24 *Let X be any topological space and let Λ be an index set. Let ψ be a bijection of Λ onto Λ. Let $\Psi(x) = x \circ \psi$ for each x in X^Λ. Then Ψ is a homeomorphism of X^Λ onto itself when X^Λ is equipped with the product topology.*

Proof For each $\lambda \in \Lambda$ and each open set $V \subset X$, let $E_{(\lambda,V)}$ be the set of all $x \in X^\Lambda$ such that $x(\lambda) \in V$.

Now $y \in \Psi^{-1}[E_{(\lambda,V)}]$ if, and only if, $y \circ \psi(\lambda) \in V$ that is $y \in E_{(\psi(\lambda),V)}$. But finite intersections of sets of the form $E_{(\lambda,V)}$ form a sub-base for the product topology. Hence Ψ is continuous. Applying the same arguments to ψ^{-1} shows that Ψ^{-1} is also continuous i.e. Ψ is a homeomorphism. □

Let us now consider one of the algebras constructed in the previous section: $B^\infty(2^{\mathbb{N}})/I_R$. Let ψ be any permutation of $\mathbb{N}$. Then this induces a bijection Ψ of $2^{\mathbb{N}}$ onto itself which we can show is a homeomorphism. But, without further information on ψ, we do not know if Ψ maps I_R to itself. Hence we do not know, in general, if Ψ will induce an automorphism of $B^\infty(2^{\mathbb{N}})/I_R = A_R$.

By slightly adapting the construction in the previous section, we can find 2^c ideals $\{I_{R\times\mathbb{N}} : R \in \mathscr{R}_0\}$ with the following properties. First, $A_{R_1\times\mathbb{N}}$ is not equivalent to $A_{R_2\times\mathbb{N}}$ unless $R_1 = R_2$. Secondly, given any infinite discrete group G, we can define an anti-action of G as permutations of $\mathbb{N}$ which induces an action of G on $B^\infty(2^{\mathbb{N}})$. This action maps each ideal $I_{R\times\mathbb{N}}$ to itself. It induces a free ergodic action of G on $B^\infty(2^{\mathbb{N}})/I_{R\times\mathbb{N}} = A_{R\times\mathbb{N}}$. We shall carry out the details of this construction below.

First we start with a feasible pair $(T,\mathbf{U})$ then we obtain a new feasible system $(T\times\mathbb{N},\mathbf{O})$ and observe that whenever R is an admissible set for $(T,\mathbf{U})$ then $R\times\mathbb{N}$ is an admissible set for $(T\times\mathbb{N},\mathbf{O})$. We then take any countably infinite group G. We define an anti-action of G on $T\times\mathbb{N}, g\to\sigma_g$ and observe that this anti-action maps each set $R\times\mathbb{N}$ to itself. For each $g\in G$ we associate a permutation of the natural numbers, ε_g. We extend this in a natural way to a permutation of $F(\mathbb{N})$, which we again denote by ε_g.

We can now define a bijection of $F(\mathbb{N})\times F(T\times\mathbb{N})$ by $\widetilde{\sigma}_g(k,K) = (\varepsilon_g(k),\sigma_g(K))$. By applying Lemma 24, to the Big Cantor space $2^{F(\mathbb{N})\times F(T\times\mathbb{N})}$ we obtain a homeomorphism of the Big Cantor space by defining $\alpha_g(x) = x\circ\widetilde{\sigma}_g$. We find that $g\to\alpha_g$ is an action of the group G which maps $X_{R\times\mathbb{N}}$ onto itself, for each R. Hence it maps each $K_{R\times\mathbb{N}}$ onto itself. Because homeomorphisms map meagre sets to meagre sets this induces an action $\overline{\alpha}$ of G as $*$-automorphisms of $C(\widehat{K_{R\times\mathbb{N}}})$. We then show that this action is (i) free and (ii) ergodic. That is (i) if p is a non-zero projection and $\overline{\alpha}_g$ acts as the identity on $pC(\widehat{K_{R\times\mathbb{N}}})$ then g is the neutral element of the group G. And (ii) if p is a projection such that $\overline{\alpha}_g(p) = p$ for every $g\in G$ then $p = 0$ or $p = 1$.

We then use the techniques of the preceding section to show that $g\to\varepsilon_g$ induces an action of G on $B^\infty(2^{\mathbb{N}})$ which maps each ideal $I_{R\times\mathbb{N}}$ onto itself and hence induces an action on $B^\infty(2^{\mathbb{N}})/I_{R\times\mathbb{N}}$. This action can be identified with the action $\overline{\alpha}$ of G as $*$-automorphisms of $C(\widehat{K_{R\times\mathbb{N}}})$. In particular, the action on $B^\infty(2^{\mathbb{N}})/I_{R\times\mathbb{N}}$ is free and ergodic.

Let $(T,\mathbf{U})$ be a feasible pair and let G be a countably infinite set. For example, we may suppose that $G = \mathbb{N}$. Then $T\times G$ may be thought of as the disjoint union of countably many copies of T, $T\times\{g_1\}, T\times\{g_2\},\ldots$.

From the definition of feasibility, $\mathbf{U} = (U_1, U_2,\ldots)$, where this is an infinite sequence of non-empty subsets of T , with $U_m\neq U_n$ whenever $m\neq n$. Let us consider the countable set $\{U_n\times\{g\} : g\in G,\ n = 1,2,\ldots\}$. Let $\mathbf{O} = (O(n))$ $(n = 1,2,\ldots)$ be an enumeration (without repetitions) of this set.

Lemma 25 *The pair* $(T\times G,\mathbf{O})$ *is feasible.*

Proof The first condition for feasibility is straightforward, so we establish the second. Take a finite subset of $T\times G$, $\{(t_j,g_j) : j = 1,2,\ldots,p\} = M$. Let $(t,g)\in(T\times G)\backslash M$.

First we observe that $U_n\times\{g\}$ is disjoint from $\{(t_j,g_j)\in M : g_j\neq g\}$ for every n.

Next we note that $t\notin\{t_j : (t_j,g_j)\in M\ \&\ g_j = g\}$. Then, because $(T,\mathbf{U})$ is a feasible pair,

$$\{n\in\mathbb{N} : t\in U_n\ \&\ U_n\cap\{t_j : (t_j,g_j)\in M\ \&\ g_j = g\} = \varnothing\}$$

is an infinite set.

Hence $\{n\in\mathbb{N} : (t,g)\in U_n\times\{g\}\ \&\ (U_n\times\{g\})\cap M = \varnothing\}$ is infinite. □

Lemma 26 *Let R be an admissible subset of T for the pair $(T,\mathbf{U})$. Then $R\times G$ is an admissible subset of $T\times G$ for the pair $(T\times G,\mathbf{O})$.*

Proof Suppose $U_n \times \{g\} \subset R\times G$. Then $U_n \subset R$. This is false. □

We may now specialise the constructions of the previous section by putting $G=\mathbb{N}$, so the feasible pair becomes $(T\times\mathbb{N},\mathbf{O})$ and the admissible sets $R\times\mathbb{N}$. If we replace $\mathbb{N}$ by any other countably infinite set we change nothing except notation.

We now let G be a countably infinite discrete group, which we will keep fixed until we specify otherwise. There is a bijection $n\to g_n$ from $\mathbb{N}$ onto G. This induces a bijection from $T\times\mathbb{N}$ onto $T\times G$. It follows that the G action on $T\times G$ defined below, can be identified with a G-action on $T\times\mathbb{N}$. A key point is that admissible sets $R\times\mathbb{N}$ are left invariant by this action of G on $T\times\mathbb{N}$; they correspond precisely to the sets $R\times G$.

If we replace G by any other countably infinite set we obtain, up to isomorphism, the same collection of 2^c commutative monotone complete C^*-algebras. We will show that G has a free ergodic action on each of these algebras. Since this holds for each choice of countably infinite group, this will show that all such groups have free ergodic actions on every one of these algebras.

For each $g\in G$ we define a bijection on $T\times G$ by $\sigma_g(t,h)=(t,g^{-1}h)$. It is clear that if $R\subset T$ then σ_g is a bijection of $R\times G$ onto itself.

It is straight forward to verify that $g\to\sigma_g$ is an injective group anti-homomorphism of G into the group of all bijections of $T\times G$. Furthermore, for each $O(n)$, $\sigma_g[O(n)]$ is $O(m)$ for some m. In particular σ_g maps the set $\{O(n):n=1,2,\dots\}$ onto itself.

We define $\varepsilon_g(n)$ to be the unique natural number such that $\sigma_g[O(n)]=O(\varepsilon_g(n))$. Then ε_g is a permutation of $\mathbb{N}$. Furthermore $g\to\varepsilon_g$ is an injective group anti-homomorphism of G into $\mathfrak{S}_\infty$, the group of all permutations of $\mathbb{N}$.

For each g, we can extend ε_g to a bijection of $F(\mathbb{N})$, which we again denote by ε_g, by setting $\varepsilon_g(\varnothing)=\varnothing$ and, when $\{n_1,\dots,n_p\}$ is a finite subset of $\mathbb{N}$, let $\varepsilon_g(\{n_1,\dots,n_p\})=\{\varepsilon_g(n_1),\dots,\varepsilon_g(n_p)\}$. Then $g\to\varepsilon_g$ is an injective group anti-homomorphism of G into the group of all bijections of $F(\mathbb{N})$.

We define π_2 to be the projection from $T\times G$ onto the second coordinate i.e. $\pi_2(t,g)=g$. We observe that for each $O(n)$, $\pi_2[O(n)]$ is a singleton. In particular, $\pi_2[\sigma_g O(n)]=g^{-1}\pi_2[O(n)]$. Let e be the neutral element of the group. Then, for $g\neq e$, $\sigma_g O(n)\neq O(n)$. Equivalently, $O(\varepsilon_g(n))\neq O(n)$. i.e. $\varepsilon_g(n)\neq n$.

We shall need the following two technical lemmas later when establishing ergodicity and freeness of the action we define.

Lemma 27 *Let (k,K) and (l,L) be in $F(\mathbb{N})\times F(T\times G)$. Then there exists $g\in G$ such that*

(i) whenever $n\in k$ then $\sigma_g O(n)\cap L=\varnothing$ and

(ii) whenever $n\in l$ then $\sigma_{g^{-1}}O(n)\cap K=\varnothing$. That is $n\in k$ implies $O(\varepsilon_{g^{-1}}(n))\cap L=\varnothing$ and $n\in l$ implies $O(\varepsilon_g(n))\cap K=\varnothing$.

Proof First we observe that π_2 maps $\{O(n):n\in k\}\cup K$ into a finite subset of G, say, $\{g_1,\dots,g_p\}$. It also maps $\{O(n):n\in l\}\cup L$ into another finite subset of G, say, $\{h_1,\dots,h_q\}$. Since G is infinite, there is a group element g which is not in the finite set $\{h_j g_i^{-1} : 1 \leq i \leq p, 1 \leq j \leq q\}$. So $g\{g_1, \ldots, g_p\} \cap \{h_1, \ldots, h_q\} = \varnothing$. Equivalently $\{g_1, \ldots, g_p\} \cap g^{-1}\{h_1, \ldots, h_q\} = \varnothing$.

Now let $n \in k$. Then $g\pi_2[O(n)] \notin \pi_2[L]$. So $\pi_2[\sigma_{g^{-1}} O(n)] \notin \pi_2[L]$. Hence $\sigma_{g^{-1}} O(n) \cap L = \varnothing$. So $O(\varepsilon_{g^{-1}}(n)) \cap L = \varnothing$.

Similarly, let $n \in l$, then $\sigma_g O(n) \cap K = \varnothing$. That is $O(\varepsilon_g(n)) \cap K = \varnothing$. $\square$

Lemma 28 *The following statement is false. There exist $g \in G \backslash \{e\}$ and $(k, K) \in F(\mathbb{N}) \times F(T \times G)$ such that for every $(l, L) \in F(\mathbb{N}) \times F(T \times G)$, whenever $k \subset l, K \subset L$ and $O_n \cap K = \varnothing$ for each $n \in l \backslash k$, then $\varepsilon_g(l) = l$ and $\sigma_g(L) = L$.*

Proof Let us assume that the given statement is true. Then, in particular, $\sigma_g(K) = K$.

Let $K = \{(t_j, g_j) : j = 1, \ldots, p\}$. Since T is infinite, we can find $t \in T \backslash \{t_j : j = 1, \ldots, p\}$. Let $L = K \cup \{(t, e)\}$. Put $l = k$. Then by the assumed statement, $\sigma_g(L) = L$. So $\sigma_g((t, e)) \in L$. That is $(t, g^{-1}) \in L$. Since $t \notin \{t_j : j = 1, \ldots, p\}$ we must have $(t, g^{-1}) = (t, e)$. But this implies that $g^{-1} = e$, which is a contradiction. $\square$

For each $g \in G$, when $K = \{(t_j, g_j) : j = 1, 2, \ldots, p\}$, we define $\widetilde{\sigma}_g(k, K)$ to be $(\varepsilon_g(k), \sigma_g[K]) = (\varepsilon_g(k), \{(t_j, g^{-1} g_j) : j = 1, 2, \ldots, p\})$. Then $g \to \widetilde{\sigma}_g$ can be shown to be an injective anti-homomorphism of G into the group of all bijections of $F(\mathbb{N}) \times F(T \times G)$.

At this point, and until further notice, we shall fix R, and work with $R \times G$, an admissible subset of $T \times G$. Later on we shall permit R to vary.

Let us recall that, for each $k \in F(\mathbb{N})$, f_k is the element of $2^{F(\mathbb{N}) \times F(T \times G)}$ which is defined as the characteristic function of

$$\{(l, L) : l \subset k, L \in F((T \times G) \backslash (R \times G)) \text{ and } O_n \cap L = \varnothing \text{ for each } n \in k \backslash l\}.$$

To avoid a forest of tiny subscripts of subscripts we shall sometimes write "$\varepsilon(g)$" instead of "ε_g".

Lemma 29 *For each $g \in G$ and for each $k \in F(\mathbb{N})$*

$$f_k \circ \widetilde{\sigma}_g = f_{\varepsilon(g^{-1})(k)}.$$

Proof By definition $f_k \circ \widetilde{\sigma}_g(l, L) = f_k(\varepsilon_g(l), \sigma_g(L))$. This takes the value 1, if, and only if, $\varepsilon_g(l) \subset k, \sigma_g(L) \in F((T \times G) \backslash (R \times G))$ and $O(n) \cap \sigma_g(L) = \varnothing$ whenever $n \in k \backslash \varepsilon_g(l)$.

Since σ_g is a bijection of $R \times G$ onto itself, $\sigma_g(L) \in F((T \times G) \backslash (R \times G))$ precisely when $L \in F((T \times G) \backslash (R \times G))$. Also $O(n) \cap \sigma_g(L) = \varnothing$ precisely when $\sigma_g^{-1}[O_n] \cap L = \varnothing$ i.e. when $O(\varepsilon(g^{-1})(n)) \cap L = \varnothing$.

We also have $n \in k \backslash \varepsilon_g(l)$ precisely when $\varepsilon(g^{-1})(n) \in \varepsilon(g^{-1})(k) \backslash l$.

So $f_k \circ \widetilde{\sigma}_g(l, L) = 1$ if, and only if, $l \subset \varepsilon(g^{-1})(k)$, $L \in F((T \times G) \backslash (R \times G))$ and $O(\varepsilon(g^{-1})(n)) \cap L = \varnothing$ whenever $\varepsilon(g^{-1})(n) \in \varepsilon(g^{-1})(k) \backslash l$.

This is equivalent to $l \subset \varepsilon(g^{-1})(k), L \in F((T \times G) \backslash (R \times G))$ and $O(p) \cap L = \varnothing$ whenever $p \in \varepsilon(g^{-1})(k) \backslash l$.

This occurs precisely when $f_{\varepsilon(g^{-1})(k)}(l, L) = 1$. Hence $f_k \circ \widetilde{\sigma}_g = f_{\varepsilon(g^{-1})(k)}$. $\square$

For each $g \in G$ let $\alpha_g : 2^{F(\mathbb{N})\times F(T\times G)} \to 2^{F(\mathbb{N})\times F(T\times G)}$ be defined by $\alpha_g(x) = x \circ \widetilde{\sigma}_g$. By Lemma 24 α_g is a homeomorphism of the Big Cantor space onto itself. Straightforward calculations show that $g \to \alpha_g$ is an injective group homomorphism of G into the group of all homeomorphisms of the Big Cantor space onto itself. By Lemma 29, $\alpha_g[X_{R\times G}] = X_{R\times G}$ for each g. So, by continuity, $\alpha_g[K_{R\times G}] = K_{R\times G}$ for each $g \in G$. Hence $g \to \alpha_g$ is an action of G as homeomorphisms of $K_{R\times G}$. Let q_R be the canonical quotient homomorphism of $B^\infty(K_{R\times G})$ onto $B^\infty(K_{R\times G})/M(K_{R\times G}) \approx C(\widehat{K_{R\times G}})$. Then, arguing as at the beginning of this section, we can define $\overline{\alpha}_g(q_R(f)) = q_R(f \circ \alpha_{g^{-1}})$ for each $g \in G$ and $f \in B^\infty(K_{R\times G})$; this gives an action of G as $*$-automorphisms on $C(\widehat{K_{R\times G}})$.

Lemma 30 *The action $g \to \overline{\alpha}_g$ is an ergodic action on $B^\infty(K_{R\times G})/M(K_{R\times G}) \approx C(\widehat{K_{R\times G}})$.*

Proof Let E be a Baire subset of $K_{R\times G}$ such that $\overline{\alpha}_g(\pi_R(\chi_E)) = q_R(\chi_E)$ for each $g \in G$. Thus $\chi_E \circ \alpha_g - \chi_E \in M(K_{R\times G})$ for each $g \in G$. Hence the symmetric difference between $\alpha_g[E]$ and E is a meagre set. But there is a unique regular open set U, such that the symmetric difference between E and U is meagre. Then the symmetric difference between $\alpha_g[U]$ and U is a meagre set. So $\alpha_g[U]\backslash clU$ is a meagre open set. By the Baire Category Theorem this set must be empty. So $\alpha_g[U] \subset clU$. Thus $\alpha_g[U] \subset int(clU) = U$. By replacing g by g^{-1} we obtain the reverse inequality. Thus $\alpha_g[U] = U$ for each $g \in G$.

Let us assume that U is neither the empty set nor the whole of $K_{R\times G}$. Then $K_{R\times G}\backslash U$ is a closed set with non-empty interior.

So, by Lemma 20, there exist (k,K) and (l,L) in $F(\mathbb{N}) \times F((T\times G)\backslash(R\times G))$ such that $\varnothing \neq E_{(l,L)} \subset K_{R\times G}\backslash U$ and $\varnothing \neq E_{(k,K)} \subset U$.

Thus $\alpha_g[E_{(l,L)}]$ is disjoint from $E_{(k,K)}$ for all $g \in G$. Equivalently, $E_{(l,L)}$ is disjoint from $\alpha_g[E_{(k,K)}]$ for all $g \in G$. □

By Lemma 27 we can find $g \in G$ such that $n \in k$ implies $O(\varepsilon_{g^{-1}}(n)) \cap L = \varnothing$ and $n \in l$ implies $O(\varepsilon_g(n)) \cap K = \varnothing$.

Put $h = \varepsilon_{g^{-1}}(k) \cup l$ and consider f_h. Now for all $p \in h\backslash l$, $p \in \varepsilon_{g^{-1}}(k)$ and so $O_p \cap L = \varnothing$. Hence $f_h(l,L) = 1$. That is $f_h \in E_{(l,L)}$.

Now consider $\alpha_g^{-1}(f_h) = f_h \circ \widetilde{\sigma}_{g^{-1}} = f_{\varepsilon(g)(h)}$ by Lemma 29. So $\alpha_g^{-1}(f_h) = f_{k\cup\varepsilon(g)(l)}$. But $q \in \varepsilon(g)(l)$ implies $O(q) \cap K = \varnothing$. So

$f_{k\cup\varepsilon(g)(l)}(k,K) = 1$. Hence $\alpha_g^{-1}(f_h) \in E_{(k,K)}$. Thus $f_h \in \alpha_g[E_{(k,K)}] \cap E_{(l,L)}$. This is a contradiction. Hence $\pi_R(\chi_E)$ is 0 or 1. In other words the action is ergodic.

Lemma 31 *The action $g \to \overline{\alpha}_g$ on $B^\infty(K_{R\times G})/M(K_{R\times G}) \approx C(\widehat{K_{R\times G}})$ is free.*

Proof Let $g \in G$ such that, for some non-zero projection $p \in C(\widehat{K_{R\times G}})$, $\overline{\alpha}_g(pz) = pz$ for each $z \in C(\widehat{K_{R\times G}})$.

Let E be a Baire set such that $\pi_R(\chi_E) = p$. Let U be the unique regular open set which differs from E only on a meagre set. Since p is not zero, U is not empty. Let Q be any clopen subset of U. Then $\pi_R(\chi_Q) \leq p$ and so $\alpha_g(Q) = Q$. By Lemma 20 we can find a non-empty clopen set $E_{(k,K)} \subset U$. (Where $K \subset (T\times G)\backslash(R\times G)$).

By hypothesis, each clopen subset of $E_{(k,K)}$ remains fixed under α_g. So, if $E_{(l,L)} \subset E_{(k,K)}$ then $E_{(l,L)} = E_{(\varepsilon(g)(l),\sigma(g)[L])}$. By Lemma 21, $l = \varepsilon_g(l)$ and $L = \sigma_g[L]$.

Suppose we are given any $(l,L) \in F(\mathbb{N}) \times F(T \times G)$, for which $k \subset l, K \subset L$ and $O_n \cap K = \varnothing$ for each $n \in l \backslash k$. Then $f_l \in E_{(l,L)} \cap E_{(k,K)}$. So, by Corollary 17, $E_{(l,L)} \subset E_{(k,K)}$. So from the preceding paragraph $l = \varepsilon_g(l)$ and $L = \sigma_g[L]$. But, by Lemma 28, this is false unless $g = e$. So the action is free. □

We now revert to our earlier notation. We use $(T \times \mathbb{N}, \mathbf{O})$ instead of $(T \times G, \mathbf{O})$ because we now wish to emphasise that we consider the actions of different groups G. The free ergodic actions we have constructed above work for any countably infinite group, giving actions on $A_{R\times\mathbb{N}}$ for each choice of R, where R is admissible for the original feasible pair $(T, \mathbf{U})$. The key point which made the above construction work was that $R \times \mathbb{N}$ was invariant for each G-action. We shall now relate our constructions to actions of quotients of the classical Cantor space.

Let Γ be the map from the Big Cantor space onto the small Cantor space, defined in the previous section by: for $x \in 2^{F(\mathbb{N})\times F(T\times\mathbb{N})}$ let $\Gamma(x)(n) = x((\{n\}, \varnothing))$ for $n = 1, 2, \ldots$. Then Γ is a continuous map from the Big Cantor space into $2^{\mathbb{N}}$, the classical Cantor space. Let $\Gamma_{R\times\mathbb{N}}$ be the restriction of Γ to $K_{R\times\mathbb{N}}$ then it is a continuous map from $K_{R\times\mathbb{N}}$ onto $2^{\mathbb{N}}$. Then $I_{R\times\mathbb{N}}$ is the σ-ideal of $B^\infty(2^{\mathbb{N}})$ consisting of all bounded Baire functions f such that $f \circ \Gamma_{R\times\mathbb{N}} \in M(K_{R\times\mathbb{N}})$. By Theorem 4 $B^\infty(2^{\mathbb{N}})/I_{R\times\mathbb{N}}$ is isomorphic to $B^\infty(K_{R\times\mathbb{N}})/M(K_{R\times\mathbb{N}}) \approx C(\widehat{K_{R\times\mathbb{N}}}) = A_{R\times\mathbb{N}}$.

We now show how the action of G can be represented on $B^\infty(K_{R\times\mathbb{N}})/M(K_{R\times\mathbb{N}})$.

We observe that, for any $x \in 2^{F(\mathbb{N})\times F(T\times\mathbb{N})}$ we have

$$\Gamma(\alpha_g(x))(n) = \alpha_g(x)(\{n\}, \varnothing) = x(\widetilde{\sigma}_g(\{n\}, \varnothing) = x(\{\varepsilon_g(n)\}, \varnothing) = \Gamma(x)(\varepsilon_g(n)).$$

For each $g \in G$ and each $y \in 2^{\mathbb{N}}$ we define $\widehat{\varepsilon}_g(y)(n) = y(\varepsilon_g(n))$. Then, by Lemma 24, $\widehat{\varepsilon}_g$ is a homeomorphism of $2^{\mathbb{N}}$ onto itself.

We have $\Gamma(\alpha_g(x)) = \widehat{\varepsilon}_g(\Gamma(x))$. That is, $\Gamma \circ \alpha_g = \widehat{\varepsilon}_g \circ \Gamma$. On taking restrictions to $K_{R\times\mathbb{N}}$ we find $\Gamma_{R\times\mathbb{N}} \circ \alpha_g = \widehat{\varepsilon}_g \circ \Gamma_{R\times\mathbb{N}}$.

Since α_g is a homeomorphism of $K_{R\times N}$ onto itself, it maps meagre sets to meagre sets. So, for $F \in B^\infty(K_{R\times\mathbb{N}}), F \in M(K_{R\times\mathbb{N}})$ if, and only if, $F \circ \alpha_g \in M(K_{R\times\mathbb{N}})$. So, for $f \in B^\infty(2^{\mathbb{N}})$,

$$f \in I_{R\times\mathbb{N}} \Longleftrightarrow f \circ \Gamma_{R\times\mathbb{N}} \in M(K_{R\times\mathbb{N}}) \Leftrightarrow f \circ \Gamma_{R\times\mathbb{N}} \circ \alpha_g \in M(K_{R\times\mathbb{N}}) \Leftrightarrow$$

$$\Leftrightarrow f \circ \widehat{\varepsilon}_g \circ \Gamma_{R\times\mathbb{N}} \in M(K_{R\times\mathbb{N}}) \Leftrightarrow f \circ \widehat{\varepsilon}_g \in I_{R\times\mathbb{N}}.$$

Hence we can define an action $g \to \widetilde{\varepsilon}_g$ by putting $\widetilde{\varepsilon}_g(f + I_{R\times\mathbb{N}}) = f \circ \widehat{\varepsilon}_{g^{-1}} + I_{R\times\mathbb{N}}$.

In the previous section, we defined a σ-homomorphism $\gamma_{R\times\mathbb{N}}$ from $B^\infty(2^{\mathbb{N}})$ to $B^\infty(K_{R\times\mathbb{N}})$ by $\gamma_{R\times\mathbb{N}}(f) = f \circ \Gamma_{R\times\mathbb{N}}$. As in Corollary 18, we let $q_{R\times\mathbb{N}}$ be the canonical quotient homomorphism of $B^\infty(K_{R\times\mathbb{N}})$ onto $B^\infty(K_{R\times\mathbb{N}})/M(K_{R\times\mathbb{N}})$. Then, see Theorem 4, $q_{R\times\mathbb{N}} \circ \gamma_{R\times\mathbb{N}}$ is a σ-homomorphism of $B^\infty(2^{\mathbb{N}})$ onto $B^\infty(K_{R\times\mathbb{N}})/M(K_{R\times\mathbb{N}})$ whose kernel is $I_{R\times\mathbb{N}}$. So $B^\infty(2^{\mathbb{N}})/I_{R\times\mathbb{N}}$ is isomorphic to $B^\infty(K_{R\times\mathbb{N}})/M(K_{R\times\mathbb{N}}) = A_{R\times\mathbb{N}}$ under an isomorphism π, where $\pi(f + I_{R\times\mathbb{N}}) = q_{R\times\mathbb{N}}(f \circ \Gamma_{R\times\mathbb{N}})$.

We have $\pi(\widetilde{\varepsilon}_g(f+I_{R\times\mathbb{N}}) = \pi(f \circ \widehat{\varepsilon}_{g^{-1}} + I_{R\times\mathbb{N}}) = q_{R\times\mathbb{N}}(f \circ \widehat{\varepsilon}_{g^{-1}} \circ \Gamma_{R\times\mathbb{N}}) = q_{R\times\mathbb{N}}(f \circ \Gamma_{R\times\mathbb{N}} \circ \alpha_{g^{-1}}) = \overline{\alpha}_g(q_{R\times\mathbb{N}}(f \circ \Gamma_{R\times\mathbb{N}})) = \overline{\alpha}_g(\pi(f+I_{R\times\mathbb{N}}))$ for all $g \in G$ and all $f \in B^\infty(2^{\mathbb{N}})$. So $\pi \circ \widetilde{\varepsilon}_g = \overline{\alpha}_g \circ \pi$ for all $g \in G$.

Hence it follows that the action $g \to \widetilde{\varepsilon}_g$ on $B^\infty(2^{\mathbb{N}})/I_{R\times\mathbb{N}}$is equivalent to the action $g \to \overline{\alpha}_g$ on $A_{R\times\mathbb{N}}$.

Let

$$\mathbf{N} : T \times \mathbb{N} \longmapsto \mathscr{P}(\mathbb{N})$$

be the map defined by

$$\mathbf{N}(t) = \{n \in \mathbb{N} : t \in O(n)\}.$$

In the statement of the following theorem and corollary, we are considering only spectroids modulo $(T \times \mathbb{N}, \mathbf{N})$. The arguments above give:

Theorem 5 *For every countably infinite group G, there exists a free ergodic action $g \to \widetilde{\varepsilon}_g$ on $B^\infty(2^{\mathbb{N}})/I_{R\times\mathbb{N}}$. In particular, this action is induced by an action of G as permutations of $\mathbb{N}$ which induces an action, as homeomorphisms, of $2^{\mathbb{N}}$ which leaves each of the σ-ideals $I_{R\times\mathbb{N}}$ invariant. Furthermore $R \times \mathbb{N}$ is in the spectroid of $A_{R\times\mathbb{N}} = B^\infty(2^{\mathbb{N}})/I_{R\times\mathbb{N}}$. That is, $R \times \mathbb{N} \in \partial(A_{R\times\mathbb{N}})$ whenever R is admissible for $(T, \mathbf{U})$.*

For the sake of definiteness, we shall suppose that $T = 2^{\mathbb{N}}$, and that $\mathbf{U}$ is an enumeration, without repetitions of the clopen subsets of $2^{\mathbb{N}}$. Let us fix $R^\flat$, a closed nowhere dense subset of $2^{\mathbb{N}}$ of cardinality c. Then $R^\flat$ is admissible for $(T, \mathbf{U})$. We may, for example, take $R^\flat$ to be the set

$$\{\mathbf{x} \in 2^{\mathbb{N}} : \mathbf{x}(2n) = 0, \text{ when } n = 1, 2, \dots\}.$$

Let $\mathscr{R}$ be the collection of all subsets of $R^\flat$ of cardinality c. So $\#\mathscr{R} = 2^c$. For each $R \in \mathscr{R}$, the set R is admissible for $(T, \mathbf{U})$ and hence $R \times \mathbb{N}$ is admissible for $(T \times \mathbb{N}, \mathbf{O})$.

Corollary 21 *There exists $\mathscr{R}_0 \subset \mathscr{R}$, such that $\#\mathscr{R}_0 = 2^c$ and, whenever R and S are distinct elements of $\mathscr{R}_0$, then $A_{R\times\mathbb{N}}$ and $A_{S\times\mathbb{N}}$ have different spectroids (modulo $(T \times \mathbb{N}, \mathbf{N})$) and $wA_{R\times\mathbb{N}} \neq wA_{S\times\mathbb{N}}$. Also, for each $R \in \mathscr{R}_0$, $R \times \mathbb{N} \in \partial(A_{R\times\mathbb{N}})$, where this spectroid is modulo $(T \times \mathbb{N}, \mathbf{N})$.*

Proof This follows from Theorem 5, Theorem 3 and Corollary 10. □

8 Conclusions

Let $\mathscr{C}$ be a collection of closed commutative $*$-subalgebras of ℓ^∞ such that each algebra A is monotone complete, is non-atomic, and, for each countably infinite discrete group G, admits a free ergodic action of G. We further suppose that the union of the spectroids of the algebras in $\mathscr{C}$ has cardinality 2^c. The existence of such a collection follows from Section 7 by using Corollary 21 and dropping to a subset if necessary. We may (and shall) also assume that distinct algebras in $\mathscr{C}$ never have the same spectroid, modulo some $(T, \mathbf{N})$. (In particular $\#\mathscr{C} = 2^c$) We fix this $(T, \mathbf{N})$.

Lemma 32 *The set $\{wA : A \in \mathscr{C}\}$ is a subset of W which has cardinality 2^c.*

Proof When A and B are distinct elements of $\mathscr{C}$ they have different spectroids and hence $wA \neq wB$. So w is an injection from $\mathscr{C}$ into W. □

Lemma 33 *Let B and A be monotone complete C^*-algebras with an isomorphism π from A onto a monotone closed subalgebra of B. Let Γ be a faithful normal conditional expectation from B onto $\pi(A)$. Then $wB = wA$.*

Proof Since π is an isomorphism it is faithful and since its range is a monotone closed subalgebra of B, it is a normal map into B. So $A \precsim B$.

Since Γ is faithful and normal, $B \precsim A$. Hence $wA = wB$. □

Let $A \in \mathscr{C}$. Let G be a countably infinite discrete group with a free ergodic action $\overline{\alpha}$ on A. Then there is more than one way of using a cross-product construction to produce a factor $A^{G\times\overline{\alpha}}$ in which there is an isomorphism π of A onto a maximal abelian subalgebra of $A^{G\times\overline{\alpha}}$ and a faithful normal conditional expectation Γ from $A^{G\times\overline{\alpha}}$ onto $\pi[A]$. So, even if the cross- products associated with A are not all known to be isomorphic they are, by Lemma 33, equivalent to each other and to A. In other words, $wA^{G\times\overline{\alpha}} = wA$. In particular, since A is not a von Neumann algebra, $wA \neq 0$. Hence $wA^{G\times\overline{\alpha}} \neq 0$ and so $A^{G\times\overline{\alpha}}$ is not a von Neumann algebra. It can be shown that each of these monotone cross-products is a small C^*-algebra. (To see this we may argue as follows. Since A is a $*$-subalgebra of ℓ^∞, it acts on ℓ^2. Hence the monotone complete tensor product $A\overline{\otimes}L(\ell^2(G))$ is completely isometric to a subsystem of $L(\ell^2 \otimes \ell^2(G))$. Since $A^{G\times\overline{\alpha}}$ is a unital $*$-subalgebra of $A\overline{\otimes}L(\ell^2(G))$, also $A^{G\times\overline{\alpha}}$ is small). [14] (See also [29]).

So we have the following:

Theorem 6 *Let the above $(T\times\mathbb{N},\mathbf{N})$ be fixed. There exists a collection $\mathscr{F}$ of small Type III factors such that each factor is wild and $\#\mathscr{F} = 2^c$. Furthermore if A and B are distinct elements of $\mathscr{F}$ then $\partial_{(T\times\mathbb{N},\mathbf{N})}(A) \neq \partial_{(T\times\mathbb{N},\mathbf{N})}(B)$ and so $wA \neq wB$. Also the union of the spectroids, (modulo $(T\times\mathbb{N},\mathbf{N})$), of the factors in $\mathscr{F}$ is of cardinality 2^c.*

For each small wild factor M its injective envelope $I(M)$ is an injective monotone complete C^*-algebra which is also a small wild factor (see for example, [13] and [15]). Furthermore the natural injection of M into $I(M)$ implies that $M \precsim I(M)$. So $wM \leq wI(M)$. In particular $\partial_{(T\times\mathbb{N},\mathbf{N})}(M) \subset \partial_{(T\times\mathbb{N},\mathbf{N})}(I(M))$. Since $wM \neq 0$ it follows that $wI(M) \neq 0$, that is, $I(M)$ is not a von Neumann algebra.

Corollary 22 *There exists a collection $\mathscr{F}_0$ of small wild injective factors such that $\#\mathscr{F}_0 = 2^c$. Furthermore if A and B are distinct elements of $\mathscr{F}_0$ then $\partial_{(T\times\mathbb{N},\mathbf{N})}(A) \neq \partial_{(T\times\mathbb{N},\mathbf{N})}(B)$ and so $wA \neq wB$. Also the union of the spectroids of the factors in $\mathscr{F}_0$ is of cardinality 2^c.*

Proof This follows from applying Theorem 3 to $\{I(A) : A \in \mathscr{F}\}$. □

It can be shown that countable sums of small injective algebras are small injective algebras. Hence w maps the small injective algebras onto a sub semi-group of W. Clearly, from Corollary 22, the cardinality of this semi-group is 2^c. So this gives a classification semigroup for injective operator systems. Also spectroids gives classification invariants for injective operator systems.

Remark 7 (Open problem) For each injective map $\mathbf{J}$ from $\mathbb{R}$ into the collection of infinite subsets of $\mathbb{N}$, there exists a corresponding spectroid $\partial_{(\mathbb{R},\mathbf{J})}(A)$ for each small monotone complete C^*-algebra A. We know that $wA = wB$ implies $\partial_{(\mathbb{R},\mathbf{J})}(A) = \partial_{(\mathbb{R},\mathbf{J})}(B)$ for every $\mathbf{J}$.

What about the converse? Suppose that $\partial_{(\mathbb{R},\mathbf{J})}(A) = \partial_{(\mathbb{R},\mathbf{J})}(B)$ whenever $\mathbf{J}$ is an injective map from the real numbers into the collection of infinite subsets of the natural numbers. Does this imply that $wA = wB$? Or are there counter examples?

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